

Lectures on Multiplicative Renormalization

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Contents

I. Multiplicative renormalization scheme	2
A. One-loop counterterms of ϕ^3 model in $d = 6$	2
B. Power counting	3
C. BPHZ	4
D. Composite operator renormalization	7
II. Multiplicative renormalization group	9
A. Solution of the RG equation	13
B. Universality	15
C. Dimensional transmutation	16
D. Logs	18
III. Effective theories	19
A. Decoupling of heavy particles	19
B. Effective theories	21
IV. High Energy vs. Statistical Physics	23
A. Critical phenomena and renormalizability	24
B. Irrelevant operators	26
V. When the power counting is mistaken: quantum anomalies	26
A. Quantum Field Theory	30
1. Path integral in nonrelativistic Quantum Mechanics	31
2. Path integral in Quantum Field Theory	36
3. Observables and Green functions	37
B. Perturbation expansion	37
1. Green functions	37
2. Connected Green functions	38
3. One-particle irreducible Green functions	39
4. One-loop effective potential of the ϕ^4 model.	40
C. Dimensional regularization	42
1. Γ -function	42
2. Loop-integrals	44

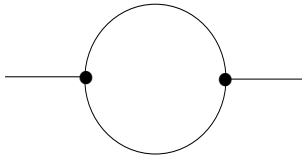


FIG. 1: One-loop correction to the propagator

I. MULTIPLICATIVE RENORMALIZATION SCHEME

The multiplicative renormalization scheme is a complete and systematic procedure to remove the UV divergences in Quantum Field Theory by exploiting the fact that the parameters of the Lagrangian are non-physical parameters only. We shall demonstrate this scheme in the framework of the ϕ^3 model in $d = 6$ dimensions, given by the Lagrangian $L = L_R + L_{CT}$ where the renormalizable Lagrangian which organizes the perturbation expansion is

$$L_R = \frac{1}{2}(\partial\phi)^2 - h\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 \quad (1)$$

and the counterterm Lagrangian, to be taken into account perturbatively is

$$L_{CT} = \frac{\delta Z}{2}(\partial\phi)^2 - \delta h\phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta g}{3!}\phi^3. \quad (2)$$

The vacuum is unstable but is still perturbatively well defined and serves as a good training ground for the isolation and removal of UV divergences. The term $\mathcal{O}(\phi)$ is included, $h \neq 0$, for the removal of the tadpoles and to assure $\langle\phi\rangle = 0$. This model displays the generic problems of renormalization at lower order of the perturbation expansion than the more realistic $d = 4$ ϕ^4 model.

A. One-loop counterterms of ϕ^3 model in $d = 6$

The bare self energy, depicted in Fig. 1, is

$$\Sigma_B(p^2) = \frac{ig^2}{2} \int_k \frac{1}{(k^2 - m^2 + i\epsilon)[(k+p)^2 - m^2 + i\epsilon]} \quad (3)$$

where $\int_k = \int d^d k / (2\pi)^d$. In dimensional regularization one performs the Wick rotation first, $k^0 \rightarrow ik^0$, $k_E^2 = k^{02} + \mathbf{k}^2$,

$$\Sigma_B(p^2) = -\frac{g^2}{2} \int_k \frac{1}{(k_E^2 + m^2)[(k+p)_E^2 + m^2]} \quad (4)$$

which does not change the divergent part of the loop-integral. We use Schwinger parameterization for the propagator,

$$\frac{1}{m^2 + k_E^2} = \int_0^\infty da e^{-a(m^2 + k_E^2)}, \quad (5)$$

yielding

$$\begin{aligned} \Sigma_B(p^2) &= -\frac{g^2}{2} \int_k \int_0^\infty da db e^{-a(k_E^2 + m^2) - b[(k+p)_E^2 + m^2]} \\ &= -\frac{g^2}{2} \int_0^\infty da db \int_k e^{-(a+b)k_E^2 - 2bkp_E - am^2 - b(p_E^2 + m^2)} \\ &= -\frac{g^2}{2} \int_0^\infty da db \int_k e^{-(a+b)k_E^2 + p_E^2 \frac{b^2}{a+b} - am^2 - b(p_E^2 + m^2)} \end{aligned} \quad (6)$$

where first the order of the integration was changed and the shift $k \rightarrow k - pb/(a+b)$ was performed. The use new variables $z = a+b$, $x = a/z$, together with the with the relations

$$\frac{b^2}{a+b} - b = (a+b) \left[\frac{b^2}{(a+b)^2} - \frac{b}{a+b} \right] = zx(x-1) \quad (7)$$

and

$$\frac{\partial[zx]}{\partial[ab]} = \det \begin{pmatrix} 1 & 1 \\ \frac{1}{z} - \frac{a}{z^2} & -\frac{a}{z^2} \end{pmatrix} = -\frac{1}{z} \quad (8)$$

yields

$$\begin{aligned} \Sigma_B(p^2) &= -\frac{g^2}{2(2\pi)^d} \int_0^1 dx \int_0^\infty dz z \int d^d k e^{-zk_E^2 - z[m^2 + p_E^2 x(1-x)]} \\ &= -\frac{g^2}{2(2\pi)^d} \int_0^1 dx \int_0^\infty dz z^{1-\frac{d}{2}} \int d^d k e^{-k_E^2 - z[m^2 + p_E^2 x(1-x)]} \\ &= -\frac{g^2 \Gamma(2 - \frac{d}{2})}{2(4\pi)^{\frac{d}{2}}} \int_0^1 dx [m^2 + p_E^2 x(1-x)]^{\frac{d}{2}-2} \end{aligned} \quad (9)$$

where Eq. (C22), the definition of the Γ -function, Eq. (C1) and Eq. (C18) were used. We have poles at $d = 4, 6, \dots$

The coupling constant is dimensional since $d = 2 + 2[\phi]$, $[\phi] = d/2 - 1$, $d = [g_n] + n[\phi]$, $[g_n] = (1 - n/2)d + n$. Therefore, we make the replacement

$$g \rightarrow g\mu^{3-d/2} = g e^{(3-d/2)\ln\mu} = g \left[1 - \frac{d-6}{4} \ln\mu^2 + \mathcal{O}((d-6)^2) \right] \quad (10)$$

and the new g introduced in this manner is dimensionless for $d \approx 6$. We have finally

$$\begin{aligned} \Sigma_B(p^2) &\approx -\frac{g^2}{2(4\pi)^3} \left(1 - \frac{d-6}{2} \ln\mu^2 \right) \left(1 - \frac{d-6}{2} \ln 4\pi \right) \left(\frac{2}{d-6} + \gamma - 1 \right) \\ &\quad \times \int_0^1 dx [m^2 + p_E^2 x(1-x)] \left(1 + \frac{d-6}{2} \ln[m^2 + p_E^2 x(1-x)] \right) \\ &\approx \frac{g^2}{(4\pi)^3} \frac{1}{d-6} \left(-m^2 + \frac{p^2}{6} \right) \\ &\quad - \frac{g^2}{2(4\pi)^3} \left[\left(m^2 - \frac{p^2}{6} \right) (\gamma - 1) + \int_0^1 dx [m^2 - p^2 x(1-x)] \ln \frac{m^2 - p^2 x(1-x)}{\ln 4\pi\mu^2} \right] \end{aligned} \quad (11)$$

where Eq. (C16) was used in the first equation.

We mention two widely applied subtraction schemes, differing in the finite parts:

- In minimal subtraction one removes the pole terms only,

$$\begin{aligned} \Sigma_{MS}(p^2) &= -\frac{g^2}{2(4\pi)^3} \left[\left(m^2 - \frac{p^2}{6} \right) (\gamma - 1) + \int_0^1 dx [m^2 - p^2 x(1-x)] \ln \frac{m^2 - p^2 x(1-x)}{\ln 4\pi\mu^2} \right], \\ \delta m^2 &= -\frac{g^2}{(4\pi)^3} m^2 \frac{1}{d-6}, \\ \delta Z &= -\frac{g^2}{6(4\pi)^3} \frac{1}{d-6}. \end{aligned} \quad (12)$$

- In BPHZ one subtracts the $\mathcal{O}(p^0)$ and $\mathcal{O}(p^2)$ terms.

The minimal subtraction of the three-particle Green function of Fig. 2 yields

$$\delta g = g^{3-\frac{d}{2}} \frac{g^3}{64\pi^3} \frac{1}{d-6}. \quad (13)$$

B. Power counting

The UV divergences with finite external leg momenta are factorized into 1PI graphs. The factorization of Green functions into the products of 1PI vertex functions and propagators is called the skeleton expansion.

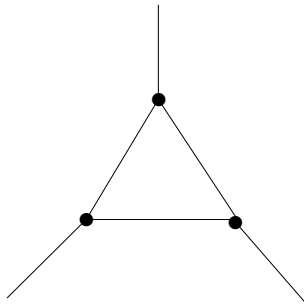


FIG. 2: One-loop correction to the coupling constant

There are several regions giving UV divergent contributions to a 1PI graph

$$I(q) = \int_{\epsilon < p < \Lambda} d^n p \frac{N_j(p, q)}{D_k(p, q)}. \quad (14)$$

The primitive, overall divergence is the contribution of the loop-integral when all components of the momenta in all internal line diverges with the same rate, $p_j \rightarrow \infty$, $p_j/p_k = \text{const}$.

$$I \approx \int_{\epsilon}^{\Lambda} dpp^{n-1+j-k}. \quad (15)$$

The degree of divergence of this contribution can be obtained on dimensional ground. Let us define the primitive degree of divergence of (15) as the energy dimension of the loop-integral for $\hbar = c = 1$,

$$\omega[I] = [I] = n + j - k. \quad (16)$$

In fact, the primitive divergence of the integral

$$I \approx \begin{cases} \left(\frac{\Lambda}{\epsilon}\right)^{\omega[I]} & \omega[I] \neq 0 \\ \ln \frac{\Lambda}{\epsilon} & \omega[I] = 0 \end{cases} \quad (17)$$

shows that the graph with $\omega[I] \geq 0$ are divergent. Notice that the divergence always turns out to be linear, $\mathcal{O}(1/(d-d_0))$, in dimensional regularization.

The degree of divergence of a graph with E external legs and V vertices in the d -dimensional ϕ^n theory is

$$\omega[G] = d + [\int] = d + (d-2)I - dV = d + E \frac{2-d}{2} + \underbrace{\left[\frac{n}{2}(d-2) - d\right]}_{-[g_n]} V \quad (18)$$

where the relation $E + 2I = nV$ has been used. This expression shows that in theories with coupling constants of non-negative energy dimensions there are finite number of Green function with primitive divergence.

The primitive divergences can always be removed by the introduction of appropriate local counterterms in the action. The fact that the divergence of the loop-integrals like Eqs. (C26)-(C28) is a polynomial in the external momentum is obvious. The general case is covered by the Weinberg theorem, stating that an 1PI graph $\Gamma[p]$ without sub-divergences remains finite after subtracting the first $\omega[I]+1$ terms of its Taylor-expansion in the external momenta,

$$\Gamma(p) = P^{[\Gamma]}(p) + \Gamma_{\text{finite}}(p) \quad (19)$$

and the local counterterms are generated by subtracting diverging polynomial $P^{[\Gamma]}(p)$ of order $\omega[\Gamma]+1$. The expansion around $p = p_0$ is called mass-independent for $p_0 = 0$.

C. BPHZ

A graphs may contain other than overall divergences, they arise from lower-order subgraphs. The removal of sub-divergences creates a problems, what can be seen easiest in dimensional regularization. In fact, the $\mathcal{O}(1/(d-d_0))$ sub-divergence of a sub-graph may multiply the $\mathcal{O}((d-d_0)^0)$ finite part of other sub-graphs. These contributions usually

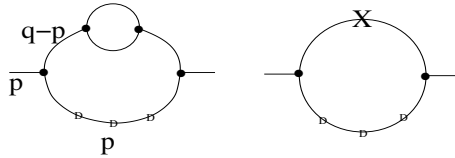


FIG. 3: One-loop correction to the propagator

include the logarithm of an momenta as in the last line of Eqs. (11). This divergent contribution, $\mathcal{O}(1/(d-d_0)) \ln p^2$ is non-polynomial in the external momentum therefore represents non-local divergence. These divergences come from its lower order sub-graphs and must be subtracted before we attack the problem of the overall divergence, at the order where they appear first as overall divergences. This can be verified by the technics, consisting of taking the derivatives of the 1PI graph with respect to the external momenta until the remaining loop-integral has negative primitive degree of divergences, like on Fig. 3 for the case of the self-energy.

The secret source of efficiency of this method is the following. Let us always subtract the overall divergences on the level of the integrands,

$$I(p) = \int dq \frac{N(q,p)}{D(q,p)} \rightarrow \int dq \left[\frac{N(q,p)}{D(q,p)} - C(q,p) \right] \quad (20)$$

and find loop-integrals with negative overall divergences, $\omega[I] \rightarrow \omega[I] - [I] - 1 = -1$. The integrand of the total graph is Λ dependent after the subtraction of the sub-divergences. But it is easy to see that for loop-integrals with negative dimension and without sub-divergences the order of the removal of the cut-off and the integration commute, $\int \lim = \lim \int$ because the integrand, being bounded by $1/|q|$, for large q converges uniformly as $\Lambda \rightarrow \infty$.

The recursive construction of local counterterms was first provided by Bogoliubov, Parasiuk, Hepp. Later Zimmermann gave an explicit solution of the procedure, the forest-formula, stating that all sub-divergences must be subtracted from a graph before calculation its overall counterterms. This procedure always yields UV finite Green functions by means of local counterterms, with number of legs given by the maximal number of legs among the UV divergent 1PI functions of the theory.

The forest formula is actually equivalent with an intuitively clear and appealing procedure: As we progress in the perturbation expansion the divergences must be subtracted by the introduction of local counterterms when they appear first as overall divergences. It is clear that this procedure identifies the same divergences than the forest-formula. What is a real tour-deforce in the construction is the calculation of the symmetry prefactors and the proof of their equivalence with those of the forest-formula.

The mass-independent BPHZ renormalization of the $d = 6$ ϕ^3 model on the one-loop order leads to the bare Lagrangian

$$L_B = L_R + L_{CT} \quad (21)$$

where the perturbation expansion is organized by the renormalized Lagrangian,

$$L_R = \frac{1}{2}(\partial\phi)^2 - h\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 \quad (22)$$

and the counterterms

$$L_{CT} = \frac{\delta Z}{2}(\partial\phi)^2 - \delta h\phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta g}{3!}\phi^3 \quad (23)$$

to be taken into account perturbatively are determined by the condition that the renormalized self energy $\Sigma_{BPHZ} = \mathcal{O}(p^4)$ and the renormalized three-point 1PI vertex is $\Gamma_{BPHZ}^{(3)}(p,q) = \mathcal{O}(p) + \mathcal{O}(q)$,

$$\begin{aligned} \delta Z &= -\frac{1}{2} \frac{\partial \Sigma_B(p^2)}{\partial p^2} \\ \delta m^2 &= \Sigma_B(0) \\ \delta g &= \Gamma_B^{(3)}(0,0). \end{aligned} \quad (24)$$

The only divergent 1PI graphs are of 2 and 3 external legs in the $d = 6$ ϕ^3 model therefore all counterterms taken into account in arbitrary high order of the loop-expansion lead to the counterterm Lagrangian (23)-(24). The

renormalized Lagrangian contains the physical, finite energy parameters of the theory and the counterterms, used in the appropriate order of the perturbation expansion, remove the divergences arising from the loop-integrals constructed by the physical, renormalized Lagrangian.

Let us compare the total, bare Lagrangian with the renormalized one. By writing the bare Lagrangian as

$$\begin{aligned} L_B &= \frac{Z}{2}(\partial\phi)^2 - (h + \delta h)\phi - \frac{m^2 + \delta m^2}{2}\phi^2 - \frac{g + \delta g}{3!}\phi^3 \\ &= \frac{1}{2}(\partial\phi_B)^2 - h_B\phi_B - \frac{m_B^2}{2}\phi_B^2 - \frac{g_B}{3!}\phi_B^3 \end{aligned} \quad (25)$$

with

$$Z = 1 + \delta Z \quad (26)$$

and

$$\phi = Z^{-\frac{1}{2}}\phi_B \quad (27)$$

we have

$$\begin{aligned} h_B &= Z^{-\frac{1}{2}}(h + \delta h) \\ m_B^2 &= Z^{-1}(m^2 + \delta m^2) \\ g_B &= Z^{-2}(g + \delta g). \end{aligned} \quad (28)$$

Therefore the renormalization, the elimination of the bare UV parameters in favor of the physical one is achieved by an appropriate rescaling of the field and the additive renormalization of the parameters. The Green functions

$$\begin{aligned} iG^{(n)}(x_1, \dots, x_n) &= \langle 0|T[\phi(x_1) \cdots \phi(x_n)]|0\rangle \\ iG_B^{(n)}(x_1, \dots, x_n) &= \langle 0|T[\phi_B(x_1) \cdots \phi_B(x_n)]|0\rangle \end{aligned} \quad (29)$$

and the 1PI vertex functions renormalize as

$$\begin{aligned} G^{(n)}(p; m^2, g, \mu) &= Z^{-n/2}G_B(p; m_B^2, g_B, \Lambda) \\ \Gamma^{(n)}(p; m^2, g, \mu) &= G^{(2)-n}G^{(n)}(p; m^2, g, \mu) = Z^{n/2}\Gamma_B(p; m_B^2, g_B, \Lambda). \end{aligned} \quad (30)$$

This scheme is called multiplicative renormalization. The finite Green functions are obtained by adjusting the parameters of the Lagrangian and performing a scale transformation on the field.

The bare perturbation expansion in the cut-off theory should be investigated in order to decide the applicability of the expansion. It is usually asymptotic at best, due to an essential singularity at $g = 0$, seen by the collapse of the vacuum when the sign of the coupling constant is flipped. The renormalized perturbation expansion is constructed from the bare one by reorganizing it according to the small, renormalized coupling constants. It is important to bear in mind that this is a formal series without any function on which it can be based upon. The counterterms, working behind the scene when they are subtracted from the integrand of the loop-integrals can be imagined as artificial modifications of the dynamics which suppress the high energy sector and render the subtracted loop-integrals finite. Thus the multiplicative renormalization expresses the Green functions at finite energy regime in terms of those of the UV.

Let us now summarize the perturbative condition of multiplicative renormalization for a theory with a single coupling constant g . The perturbation series for an observable is

$$\mathcal{O} = \sum_n g^n I_n. \quad (31)$$

The relation among the energy dimensions,

$$[I_n] = [\mathcal{O}] - n[g] \quad (32)$$

suggests to separate three cases:

1. $[g] < 0$: infinitely many divergent order in the perturbation expansion, the BPHZ construction needs infinitely many counterterms in L_{CT} , the theory non-renormalizable.

2. $[g] = 0$: finite number of divergent order in the perturbation expansion but infinitely many divergent graphs. There are finite number of counterterms in L_{CT} , the theory is renormalizable.
3. $[g] > 0$: finite divergent graphs, the theory is super-renormalizable.

This classification is valid for particles with spin less than one only. The propagator for massive vector field has slower decay for large momenta and the mass parameter, despite having positive dimensions, is non-renormalizable.

The power counting (17) shows that graphs with good UV behavior are IR enhanced. The IR divergences are not related to local counterterms and have to be eliminated in one way or other for the theory to make sense. In the case of massless super-renormalizable theories where the IR divergences appear at arbitrary high order in the naive perturbation series one has to perform a partial resummation of the perturbation expansion to eliminate these divergences.

The counterterm replaces the coupling constant in the Lagrangian thus it has the same dimension. Therefore the counterterms of renormalizable or super-renormalizable parameters have logarithmic or power divergences. Thus super-renormalizable parameters need more precise fine-tuning during renormalization than the renormalizable coupling constants.

A relation among observables is called natural if it holds for any order one values of the dimensionless coupling constants of the theory. This name reflects the belief that all dimensionless numbers ought to be order one in a reasonable theory. There are two important consequences of this assumption. One is that super-renormalizable parameters are not natural because they must be fine-tuned with a precision which increases fast with the cut-off. They become natural if there is a symmetry which protects against dressing. This can be a chiral or an internal symmetry for fermion or boson masses, respectively and the spontaneous breaking of these symmetries becomes a unique natural mechanism to generate natural masses at low energies. Another point is that all renormalizable couplings of the low energy theory which have no reason to be suppressed by the heavy particles should be finite, i.e. order one when expressed in natural units. The best known exception to this rule is the cosmological constant.

The perturbative elimination of the UV divergences from the 1PI vertex functions at any order of the perturbation expansion is actually not enough for the true renormalization of a theory if the latter is meant convergence of the physical content of the model during the removal of the cut-off. Let us consider QED with two adjustable parameters, e_B and m_B in its Lagrangian. The theory is fixed by the two renormalization conditions, say the Thomson-scattering cross section and an atomic energy level,

$$\begin{aligned} \sigma_{\text{TH}} &= F_1(e_B, m_B, \Lambda) \\ E_1 - E_0 &= F_2(e_B, m_B, \Lambda) \end{aligned} \quad (33)$$

The left hand sides of these equations contain experimental numbers, the right hand sides display theoretical calculations for the observables in question. The solution of the renormalization conditions, the inversion of these equations provides us with the renormalized trajectory of QED, the curve $e_B(\Lambda), m_B(\Lambda)$ in the parameter space of the theory where each point of the curve corresponds to a theory which reproduces the two experimental data. The theory is called renormalizable if every observable converges along the renormalized trajectory. But the right hand side of the renormalization conditions are complicated, non-linear functions of the parameters e_B and m_B and the system of equations may not be invertible. For low enough cut-off the loop-corrections are weak and the solution should exist. But the increase of the cut-off may render the system of equations insoluble. Usually a divergence in the coupling constant signals the lowest value of the cut-off where the renormalization conditions are noninvertible. This value of the cut-off is called UV Landau-pole and it indicates that there is no way to reproduce the desired low energy physics with the type of local Lagrangian considered if it is extended beyond the Landau-pole.

The renormalizable coupling constants of a certain particle family provide the free parameters for the finite energy dynamics of these particles which can be realized by local interactions. There are infinitely many possible vertices the particles realize but all of them are expressible in terms of the free renormalizable parameters.

D. Composite operator renormalization

The Green function with a composite operator $\phi^n(x)$ inserted contains new UV divergences, generated by the limiting procedure $\prod_{j=1}^n \lim_{x_j \rightarrow x} \phi(x_j) = \phi^n(x)$. The momentum in the Green function with small separation in space-time is large and the un-suppressed loop-integration generates large contributions to the Green function. The goal of the composite operator renormalization is to make subtractions which render the Green function with the composite operator insertions finite.

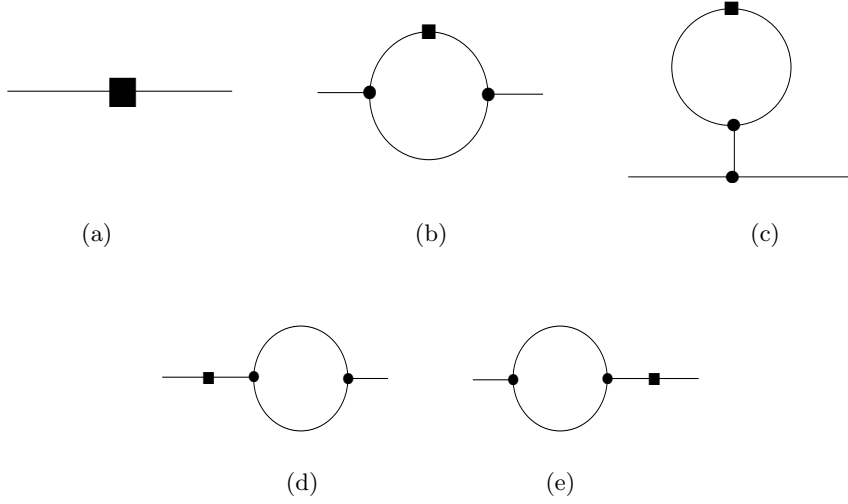


FIG. 4: One-loop corrections to the renormalization of the composite operator ϕ^2 . The square stands for the insertion of $\frac{1}{2}\phi^2$

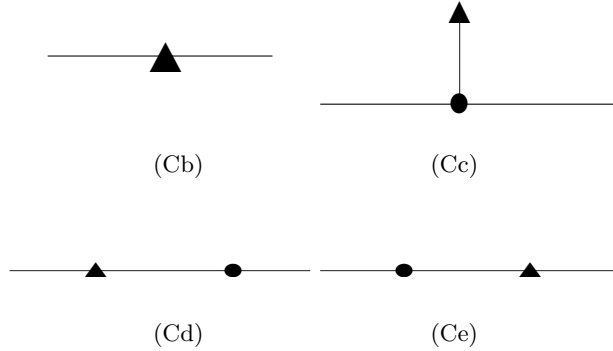


FIG. 5: One-loop counterterms, denoted by triangles, for the composite operator ϕ^2 .

We shall take $\phi^2(x)$ as the composite operator and consider the Green functions $\langle 0|T[\phi(p_1)\phi(p_2)\phi^2(-p_1-p_2)/2]|0\rangle$. The graphs shown in Fig. 4 contribute in the one-loop order to this Green function. The graph (a),

$$G_a = \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \quad (34)$$

is of tree-level and finite. The divergence of graphs (d) and (e) is removed by the usual counterterms δm^2 and δZ , shown in Fig. 4 (a)-(b). Let us now consider the graphs (b) and (c) after Wick-rotation,

$$\begin{aligned} G_b &= -\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} g^2 \mu^{6-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)[(k - p_1)^2 - m^2][(k + p_2)^2 - m^2]} \\ &= -\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{g^2}{64\pi^3} \Gamma\left(3 - \frac{d}{2}\right) \int_0^1 dx \int_0^{1-x} dy \left(\frac{m^2 - (p_1^2 y + p_2^2 x)(1-x-y) - (p_1 + p_2)^2 xy}{4\pi\mu^2} \right)^{\frac{d}{2}-3} \\ G_c &= -i \frac{g^2 \mu^{6-d}}{2(p_1^2 - m^2)(p_2^2 - m^2)[(p_1 + p_2)^2 - m^2]} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)[(k + p_1 + p_2)^2 - m^2]} \\ &= \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)[(p_1 + p_2)^2 - m^2]} \frac{g^2}{128\pi^3} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx \frac{[m^2 - (p_1 + p_2)^2 x(1-x)]^{d/2-2}}{(4\pi\mu^2)^{d/2-3}}. \end{aligned} \quad (35)$$

The minimal subtraction consists of subtracting the contributions

$$\begin{aligned} CG_b &= -\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{g^2}{64\pi^3(d-6)} \\ CG_c &= -\frac{g\mu^{d-d/2}}{(p_1^2 - m^2)(p_2^2 - m^2)[(p_1 + p_2)^2 - m^2]} \frac{g\mu^{d-d/2}}{64\pi^3(d-6)} \left[m^2 - \frac{1}{6}(p_1 + p_2)^2 \right]. \end{aligned} \quad (36)$$

The counterterms generated by these subtractions are

$$\frac{1}{2}\phi_R^2 = \left[1 + \underbrace{\frac{g^2}{64\pi^3(d-6)}}_{CG_b}\right] \frac{1}{2}\phi^2 + \underbrace{\frac{g\mu^{d-d/2}}{64\pi^3(d-6)} \left[m^2 + \frac{1}{6}\square\right]}_{CG_c} \phi = \frac{Z_a}{2}\phi^2 + \mu^{d/2-3}Z_b m^2\phi + \mu^{d/2-3}Z_c\square\phi. \quad (37)$$

We are actually renormalizing here the generating functional

$$e^{iW[j_e, j_c]} = \int D[\phi] e^{iS[\phi] + i j_e \cdot \phi + i j_c \cdot \mathcal{O}} \quad (38)$$

for the Green functions for the composite and elementary operators, \mathcal{O} and ϕ , respectively, in arbitrary order of the perturbation expansion. This program is possible for renormalizable composite operators only and is equivalent with the renormalization of the generating functional for the Green functions for the elementary operators in the presence of space-time depending renormalizable parameters in the Lagrangian.

For space-time independent sources j_c this generating functional is given by the vacuum-to-vacuum transition amplitude as the function of the renormalizable coupling constants. Therefore no new UV divergences appear compared to the usual renormalization process when the composite operator has vanishing momentum. When the source j_c has space-time dependence with finite, $\mathcal{O}(\Lambda^0)$ wave vectors then new counterterms are needed. These modify only the definition of the composite operators without effecting the Green functions for the elementary field. They require the insertion of several other operators along with the original composite one as in Eq. (37). Therefore should not be included in the counterterm Lagrangian rather in the j_c -dependent part in the exponent by writing

$$e^{iW[j_e, j_c]} = \int D[\phi] e^{iS[\phi] + i j_e \cdot \phi + i j_c \cdot \mathcal{O}_R} \quad (39)$$

with

$$\mathcal{O}_{AR} = \sum_B Z_{AB} \mathcal{O}_B \quad (40)$$

for a set of elementary and composite operators $\{\mathcal{O}_A\} = \{\phi, \frac{1}{2}\phi^2, \square\phi, \dots\}$.

The general rule for this operator mixing is that lower dimensional operators mix in only. Let us consider a 1PI graph G of the form gI , the product of some coupling constants and a loop-integral I whose counterterm, $Z_{AB}\mathcal{O}_B$, contains D space-time derivatives. The counterterm has the same dimension than the graph it belongs to,

$$[G] = [g] + [I] = [Z_{AB}] + D \quad (41)$$

The degree of divergence is at least the order of the momentum in the counterterm ($[I] > D$ when the mass carries part of the dimension of the loop-integral), thus

$$D \leq [I] = [Z_{AB}] + D - [g] \leq [Z_{AB}] + D, \quad (42)$$

$[Z_{AB}] \geq 0$ and $[\mathcal{O}_B] \leq [\mathcal{O}_A]$. The missing dimensions $[\mathcal{O}_A] - [\mathcal{O}_B]$ are provided by the space-time derivatives and/or mass factors.

II. MULTIPLICATIVE RENORMALIZATION GROUP

It is essential to realize that the renormalized and the bare parameters of the theory characterize the dynamics at different scales. The renormalized parameters arise from some renormalization conditions imposed on observables at the physical, observational scale. The bare parameters of the cut-off theory belong to the dynamics at the cut-off. To see this imagine the Hamiltonian of a solid state lattice. Its parameters, the excitation energies of molecules and the matrix elements of observables calculated by means of the molecular orbits, refer to the dynamics of the ions and the hopping of the electrons, features of the dynamics of the lattice spacing, the cut-off of this Hamiltonian. In general, the mathematics of the theory is founded defined on the bare, cut-off level and the parameters appearing in these expressions are the natural parameters of the cut-off scale.

We would like now to argue that the renormalized or the bare Green functions show the dynamics at finite or at the cut-off scale. The renormalized and the bare Green functions are proportional to each other according to the first line of Eqs. (30) therefore one has to be more precise. Let us rather turn to the structure of the Green functions, the way how they are made up from the parameters of the theory.

The divergent loop-integrals are dominated by high energy modes. The integrand might converge to zero for large momenta but primitive divergences arise from the integration over large momenta. Thus the divergent loop-integrals are estimated by the power counting and are constructed in terms of the bare parameters. When counterterms are subtracted from the integrands as in Eq. (20) then the resulting loop-integrals are finite. The UV finite loop-integrals can be subject of dimensional analysis which predicts that they are dominated by modes around the mass scales of the renormalized Lagrangian which provides the mass scales for the propagators.

This argument shows that the multiplicative renormalization, Eqs. (30), expresses the relation among Green functions in the UV and finite energy regimes. This is a diverging distance between the scales realized by subtracting divergent quantities. A finite change of the subtraction scale generates similar relations, called multiplicative renormalization group which can be realized by finite subtractions.

Let us start with the introduction of the renormalized, physical parameters, the running coupling constant $g(\mu)$ and the running mass $m^2(\mu)$, defined by renormalization conditions

$$\begin{aligned} g^2(\mu) &= \Gamma^{(3)}(\mu) \\ m^2(\mu) &= \mu^2 - G^{-1}(p^2 = \mu^2) \end{aligned} \quad (43)$$

imposed at some scale in the ϕ^3 theory in $d = 6$. Here $\Gamma^{(3)}(\mu)$ stands for the value of the 1PI three-point function $\Gamma^{(4)}(p_1, p_2, p_3)$, evaluated at $p_j^2 = \mu^2$, $p_j p_k = -\mu^2/2$ and $G(p^2)$ denotes the renormalized propagator. It is clear from the construction that $m^2(\mu)$ and $g(\mu)$ correspond to the dynamics of scale μ and we can choose the counterterms in such a manner that $m^2(\mu)$ and $g(\mu)$ becomes the mass and coupling constant of the renormalized Lagrangian,

$$L_R = \frac{1}{2}(\partial\phi)^2 - h\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3. \quad (44)$$

Once the renormalized and counterterm parameters are known we have the full, bare Lagrangian

$$L_B = \frac{1}{2}(\partial\phi_B)^2 - h_B\phi_B - \frac{m_B^2}{2}\phi_B^2 - \frac{g_B}{3!}\phi_B^3, \quad (45)$$

with parameters which characterize the dynamics at the scale of the cut-off.

Eqs. (30) describe the change of the Green functions when the parameterization changes from the cut-off to finite energy scale. Similar relations exist for a finite change of scale. In fact, let us compare two renormalized perturbation expansions, based on the scales μ_1 and μ_2 of the same bare theory. The relations (30) remain valid for both sets of renormalized Green functions therefore we have

$$\begin{aligned} G^{(n)}(p; m^2(\mu_1), g(\mu_1), \mu_1) &= z^{-n/2} G^{(n)}(p; m^2(\mu_2), g(\mu_2), \mu_2) \\ \Gamma^{(n)}(p; m^2(\mu_1), g(\mu_1), \mu_1) &= z^{n/2} \Gamma^{(n)}(p; m^2(\mu_2), g(\mu_2), \mu_2) \end{aligned} \quad (46)$$

with $z = Z(\mu_2)/Z(\mu_1)$. The change of subtraction scale is therefore a modification of the split of the total, bare Lagrangian into renormalized and counterterm part,

$$L_R \rightarrow L_R + \Delta L, \quad L_{CT} \rightarrow L_{CT} - \Delta L. \quad (47)$$

The possible splits form an n -dimensional space where n is the number of the renormalizable parameters of the theory and only one out of these n possibilities corresponds to a change of scale. The other modifications of the splitting could in principle be used to follow multi-scale dependence.

Eq. (46) holds for the full solution only. Truncation errors arise from our failure to find the exact relations between high energy (bare) and low energy (renormalized) parameters. But in practice we do not really have to bridge this large number of order of magnitudes. All we need usually is the prediction of the theory which was fixed μ_1 at another scale μ_2 which is not too far from μ_1 . We shall establish now the flow of the renormalized parameters of the theory between these finite scales, without referring to the distant, bare dynamics.

The key point is that any non-trivial scale dependence comes from interactions therefore $(\mu_2 - \mu_1)/\mu_1$ can play the role of small parameter in this calculation. One expects therefore, that the integration of the evolution equations, derived approximately in the limit $\mu_2 \rightarrow \mu_1$ performs a partial resummation of the perturbation expansion.

The renormalization group equation expresses the independence of the bare theory from the choice of our subtraction scale. It reads as

$$\begin{aligned} \mu \frac{d}{d\mu} \mathcal{O} &= \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_m m^2 \frac{\partial}{\partial m^2} \right] \mathcal{O} = 0 \\ \beta &= \mu \frac{dg(\mu)}{d\mu} \\ -\gamma_m &= \mu \frac{dm(\mu)}{d\mu} \end{aligned} \quad (48)$$

for the $d = 6$, ϕ^3 theory. It is the multiplicative renormalization scheme which assures that the change of subtraction scale can be incorporated into the adjustment of the renormalizable parameters of the theory. Thus this equation is only valid for renormalizable theories and far from the cut-off.

The coefficient functions can be obtained from the expressions relating the bare and renormalized quantities, eg.

$$\mu \frac{\partial g_B}{\partial \mu} = \mu \frac{\partial m_B^2}{\partial \mu} = 0 \quad (49)$$

It is naturally simplest to use mass-independent renormalization schemes at this point. The expressions

$$\begin{aligned} g_B &= \mu^{3-\frac{d}{2}} g \left[1 + \frac{3g^2}{256\pi^3(d-6)} + \mathcal{O}(g^4) \right] \\ m_B^2 &= m^2 \left[1 + \frac{5g^2}{384\pi^3(d-6)} + \mathcal{O}(g^4) \right] \end{aligned} \quad (50)$$

yield the coefficient functions for the minimal subtraction parameters,

$$\begin{aligned} 0 &= \mu \frac{\partial g_B}{\partial \mu} = \frac{6-d}{2} \mu^{3-\frac{d}{2}} g \left[1 + \frac{3g^2}{256\pi^3(d-6)} \right] + \beta \mu^{3-\frac{d}{2}} \left[1 + \frac{9g^2}{256\pi^3(d-6)} \right] + \mathcal{O}(g^5) \\ \beta &= \frac{d-6}{2} g \left[1 + \frac{3g^2}{256\pi^3(d-6)} \right] \left[1 - \frac{9g^2}{256\pi^3(d-6)} \right] + \mathcal{O}(g^5) \\ &= \frac{d-6}{2} g - \frac{3g^3}{256\pi^3} + \mathcal{O}(g^5) \end{aligned} \quad (51)$$

and

$$\begin{aligned} 0 &= \mu \frac{\partial m_B^2}{\partial \mu} = -\gamma_m [1 + \mathcal{O}(g^2)] + m^2 \beta \left[\frac{5g^2}{192\pi^3(d-6)} + \mathcal{O}(g^3) \right] \\ \gamma_m &= m^2 \beta \left[\frac{5g^2}{192\pi^3(d-6)} + \mathcal{O}(g^3) \right] [1 + \mathcal{O}(g^2)] \\ &= m^2 \frac{5g^2}{256\pi^3} + \mathcal{O}(g^3). \end{aligned} \quad (52)$$

Let us summarize the different options we have in the case of the $d = 4$ ϕ^4 theory where the 1PI Green functions satisfy

$$\Gamma^{(n)}(p; m^2, g, \mu) = Z^{n/2} \Gamma_B^{(n)}(p; m_B^2, g_B, \Lambda) + \mathcal{O}\left(\frac{\mu^2}{\Lambda^2}\right) + \mathcal{O}\left(\frac{p^2}{\Lambda^2}\right). \quad (53)$$

We adopt the renormalization conditions

$$\begin{aligned} \Gamma^{(2)}(0; m^2, g, \mu) &= m^2 \\ \partial_p^2 \Gamma^{(2)}(0; m^2, g, \mu) &= 1 \\ \Gamma^{(4)}(0; m^2, g, \mu) &= g \\ \Gamma^{(2,1)}(0, 0; m^2, g, \mu) &= 1 \end{aligned} \quad (54)$$

to define the running parameters, extracted from observables. $\Gamma^{(m,n)}$ denote the 1PI vertex function for m elementary field variables ϕ and n composite operators ϕ^2 .

The bare RG equation is obtained by keeping the renormalized theory fixed,

$$\begin{aligned} 0 &= \Lambda \frac{d}{d\Lambda} \Gamma^{(n)}(p; m^2, g, \mu) = \Lambda \frac{d}{d\Lambda} \left[Z^{n/2}(m_B^2, g_B, \Lambda) \Gamma_B^{(n)}(p; m_B^2, g_B, \Lambda) \right] \\ 0 &= \left[\mu \frac{\partial}{\partial \Lambda} + \beta_B \frac{\partial}{\partial g_B} - \gamma_{mB} m_B^2 \frac{\partial}{\partial m_B^2} + \frac{n}{2} \gamma_B \right] \Gamma_B^{(n)}(p; m_B^2, g_B, \Lambda) \end{aligned} \quad (55)$$

where

$$\begin{aligned}
\beta_B &= \Lambda \frac{\partial g_B}{\partial \Lambda} \Big|_{m^2, g, \mu} \\
-\gamma_{mB} &= \frac{\Lambda}{m_B^2} \frac{\partial m_B^2}{\partial \Lambda} \Big|_{m^2, g, \mu} \\
\gamma_B &= \frac{\Lambda}{Z} \frac{\partial Z}{\partial \Lambda} \Big|_{m^2, g, \mu}
\end{aligned} \tag{56}$$

The 'renormalized' RG equation are obtained by keeping the bare theory fixed,

$$\begin{aligned}
0 &= \mu \frac{d}{d\mu} \Gamma_B^{(n)}(p; m_B^2, g_B, \Lambda) = \mu \frac{d}{d\mu} \left[Z^{-n/2}(m^2, g, \mu) \Gamma^{(n)}(p; m^2, g, \mu) \right] \\
0 &= \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_m m^2 \frac{\partial}{\partial m^2} - \frac{n}{2} \gamma \right] \Gamma^{(n)}(p; m^2, g, \mu)
\end{aligned} \tag{57}$$

with

$$\begin{aligned}
\beta &= \mu \frac{\partial g}{\partial \mu} \Big|_{m_B^2, g_B, \Lambda} \\
-\gamma_m &= \frac{\mu}{m^2} \frac{\partial m^2}{\partial \mu} \Big|_{m_B^2, g_B, \Lambda} \\
\gamma &= \frac{\mu}{Z} \frac{\partial Z}{\partial \mu} \Big|_{m_B^2, g_B, \Lambda}
\end{aligned} \tag{58}$$

Finally, the Callan-Symanzik equation is found when the bare theory is changed in its mass only, ie. Λ and g_B kept fixed:

$$\begin{aligned}
m \frac{d}{dm} \Gamma_B^{(n)}(p; m_B^2, g_B, \Lambda) &= m \frac{d}{dm} \left[Z^{-n/2}(g, \mu) \Gamma^{(n)}(p; g, \mu) \right] = m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda} \frac{\partial}{\partial m_B^2} \Gamma_B^{(n)}(p; g_B, \Lambda) \\
&= m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda} \Gamma_B^{(n,1)}(p, 0; g_B, \Lambda)
\end{aligned} \tag{59}$$

The relation

$$\frac{\partial}{\partial m^2} \frac{1}{p^2 - m^2} = \frac{1}{p^2 - m^2} \cdot \frac{1}{p^2 - m^2} \tag{60}$$

was used to arrive at the second equation in (59). The chain rule now yields

$$\begin{aligned}
\left[m \frac{\partial}{\partial m} + \beta_{CS} \frac{\partial}{\partial g} - \frac{n}{2} \gamma_{CS} \right] \Gamma^{(n)}(p; g, \mu) &= Z^{n/2}(g, \mu) m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda} \Gamma_B^{(n,1)}(p, 0; g_B, \Lambda) \\
&= Z_{\phi^2}^{-1/2}(g, \mu) m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda} \Gamma^{(n,1)}(p, 0; g, \mu) \\
\beta_{CS} &= m \frac{\partial g}{\partial m} \Big|_{g_B, \Lambda} \\
\gamma_{CS} &= \frac{1}{Z} m \frac{\partial Z}{\partial m} \Big|_{g_B, \Lambda}
\end{aligned} \tag{61}$$

by ignoring the operator mixing at the right hand side of the renormalization group equation. To find the coefficient on the right hand side we consider the case $n = 2$,

$$\begin{aligned}
\left[m \frac{\partial}{\partial m} + \beta_{CS} \frac{\partial}{\partial g} - \gamma_{CS} \right] \underbrace{\Gamma^{(2)}(0; g, \mu)}_{m^2} &= Z_{\phi^2}^{-1/2}(g, \mu) m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda} \underbrace{\Gamma^{(n,1)}(0, 0; g, \mu)}_1 \\
(2 - \gamma_{CS}) m^2 &= Z_{\phi^2}^{-1/2}(g, \mu) m \frac{\partial m_B^2}{\partial m} \Big|_{g_B, \Lambda}
\end{aligned} \tag{62}$$

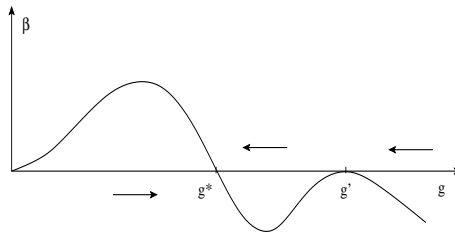


FIG. 6: The β -function of a model with several scaling regimes. The arrows show the IR direction. The fixed points are the origin (UV), g^* (IR stable) and g' (IR unstable or stable from the left or right, respectively).

and can write

$$\left[m \frac{\partial}{\partial m} + \beta_{CS} \frac{\partial}{\partial g} - \frac{n}{2} \gamma_{CS} \right] \Gamma^{(n)}(p; g, \mu) = (2 - \gamma_{CS}) m^2 \Gamma^{(n,1)}(p, 0; g, \mu). \quad (63)$$

Contrary to the previous two renormalization group equation, the Callan-Symanzik equation relates different theories, the dynamics changes during the evolution. The right hand side of the renormalization group equation, $\Gamma^{(n,1)}(p, 0; g, \mu)$ contains one more propagator than $\Gamma^{(n)}(p; g, \mu)$ on the left hand side and is suppressed by $1/p^2$ according to a sharper version of Weinberg theorem. This suppression justifies the neglect of the right hand side for large Euclidean p^2 where a renormalization group like homogeneous equation is recovered for $\Gamma^{(n)}(p; g, \mu)$.

The renormalization showed a new face in this section. It was introduced before to handle the UV divergences and it turned out to be useful to trace scale dependence in finite quantities. This latter application is actually more general and fundamental, as we shall see in the discussion of the functional form of the RG. Why did the UV divergence play such an exclusive role for long time in the renormalization group? We shall argue in section V that the UV divergences and the renormalizable parameters are indeed related except a class of important exception.

A. Solution of the RG equation

The finite change of subtraction scale generates the renormalization group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_m m^2 \frac{\partial}{\partial m^2} + \frac{n}{2} \gamma \right] G^{(n)}(p; m^2, g, \mu) = 0 \quad (64)$$

for the Green functions. To find its solution first we integrate the coefficient functions,

$$\int \frac{d\mu}{\mu} = \int \frac{dg}{\beta} \\ \mu' = \mu e^{\int_g^{g'} \frac{dg}{\beta}} \quad (65)$$

The root of the β -function is called fixed point because the coupling constant becomes scale independent there. The free massless theory is scale invariant, thus we always have the Gaussian fixed point, $g = 0, \beta(0) = 0$ which is IR or UV stable for $\beta'(0) > 0$ or $\beta'(0) < 0$ (asymptotic freedom), respectively. It may happens that the beta function develops another, non-Gaussian fixed-point, $\beta(g_{cr}) = 0$, with $\beta(g) \neq 0$ for $0 < g < g_{cr}$. This non-Gaussian fixed-point is IR or UV stable for $\beta(g) < 0$ or $\beta(g) > 0$ for $0 < g < g_{cr}$, respectively. There might be several scaling regimes, as shown in Fig. 6.

The β -function

$$\beta = \beta_1 g^{n_\beta} (1 + \mathcal{O}(g^2)) \quad (66)$$

gives in leading order

$$\mu' = \mu e^{\int_g^{g'} \frac{dg}{\beta_1 g^{n_\beta}}} = \mu e^{\frac{1}{(n_\beta-1)\beta_1 g^{n_\beta-1}} - \frac{1}{(n_\beta-1)\beta_1 g'^{n_\beta-1}}} \\ \frac{1}{\beta_1 g^{n_\beta-1}} = \frac{1}{\beta_1 g'^{n_\beta-1}} - \frac{n_\beta-1}{2} \ln \frac{\mu'^2}{\mu^2} \\ g^{n_\beta-1} = \frac{g'^{n_\beta-1}}{1 - \beta_1 \frac{n_\beta-1}{2} g'^{n_\beta-1} \ln \frac{\mu'^2}{\mu^2}} \quad (67)$$

as long as $n_\beta \neq 1$.

The integration of the γ_m coefficient function yields the running mass

$$\begin{aligned} \int \frac{dm^2}{m^2} &= - \int \frac{d\mu}{\mu} \gamma_m = - \int dg \frac{\gamma_m}{\beta} \\ m'^2 &= m^2 e^{-\int_g^{g'} dg \frac{\gamma_m}{\beta}}. \end{aligned} \quad (68)$$

The finite wave function renormalization constant of Eqs. (46), arising from the finite change of subtraction point, is given by

$$\begin{aligned} \int \frac{dz}{z} &= \int \frac{d\mu}{\mu} \gamma = \int dg \frac{\gamma}{\beta} \\ z &= e^{\int_g^{g'} dg \frac{\gamma}{\beta}} \end{aligned} \quad (69)$$

The final solution of the renormalization group equation (64) is

$$G^{(n)}(p; m'^2, g', \mu') = e^{-\frac{n}{2} \int_g^{g'} dg \frac{\gamma}{\beta}} G^{(n)}(p; m^2, g, \mu). \quad (70)$$

Eq. (70) relates the Green functions at the same momenta but different renormalized parameters. A physically more motivated question is to compare the same Green function at different momenta. The response to this question will show more explicitly the structure of the Green functions in terms of the renormalized parameters, alluded to at the beginning of this section. We transform Eq. (70) into the desired form by means of a simple dimensional argument in the special case where all momentum component are rescaled by the same factor, $p \rightarrow \kappa p$. In fact, we write $\mu' = \kappa \mu$ and find from Eq. (70)

$$G^{(n)}(\kappa p; m^2(\kappa \mu), g(\kappa \mu), \kappa \mu) = z^{-n/2} (\kappa \mu \leftarrow \mu) G^{(n)}(\kappa p; m^2(\mu), g(\mu), \mu) \quad (71)$$

and

$$G^{(n)}(\kappa p; m^2(\mu), g(\mu), \mu) = z^{n/2} (\kappa \mu \leftarrow \mu) G^{(n)}(\kappa p; m^2(\kappa \mu), g(\kappa \mu), \kappa \mu) \quad (72)$$

In the limit $\kappa \rightarrow \infty$ the mass is negligible and

$$G^{(n)}(\kappa p; m^2(\mu), g(\mu), \mu) = z^{n/2} (\kappa \mu \leftarrow \mu) G^{(n)}(\kappa p; 0, g(\kappa \mu), \kappa \mu). \quad (73)$$

Simple dimensional analysis now gives

$$G^{(n)}(\kappa p; m^2(\mu), g(\mu), \mu) = \kappa^{[G]} z^{n/2} (\kappa \mu \leftarrow \mu) G^{(n)}(p; 0, g(\kappa \mu), \mu), \quad (74)$$

showing that $g(\kappa \mu)$ is indeed the coupling strength which characterizes the scale $\kappa \mu$ and z gives a quantum fluctuation generated correction to the classical scaling laws and introduces anomalous dimension.

We reduced the high energy limit to the massless, IR limit by means of dimensional analysis. But it may happens that some IR singularity at $m = 0$ creates the impression that the theory has singular high energy limit. This illusion arise from the IR singularities of ill chosen counterterms. One can prove that one can always choose counterterms which are polynomial in the mass and the massless limit is safe.

Similar dimensional argument, carried out at a non-Gaussian fixed-point gives

$$\begin{aligned} G^{(n)}(\kappa p; m^2(\mu), g(\mu), \mu) &= \kappa^{[G]} e^{\frac{n}{2} \int_g^{g'} dg \frac{\gamma}{\beta}} G^{(n)}(p; 0, g(\kappa \mu), \mu) \\ &\approx \kappa^{[G]} \left(\frac{\mu'}{\mu} \right)^{\frac{n}{2} \gamma^*} G^{(n)}(p; 0, g(\kappa \mu), \mu) \\ &\approx \kappa^{[G] + \frac{n}{2} \gamma^*} G^{(n)}(p; 0, g(\kappa \mu), \mu) \end{aligned} \quad (75)$$

and shows potentially strong anomalous dimension.

B. Universality

Let us consider two subtraction schemes giving,

$$G_j^{(n)}(p; m_j^2, g_j, \mu) = Z_j^{-n/2} G_B(p; m_B^2, g_B, \Lambda) \quad (76)$$

$j = 1, 2$, both of them are supposed to be mass-independent for simplicity. We assume the perturbative forms

$$\begin{aligned} \beta_j(g) &= g^{n_\beta} (A_{j1} + A_{j2}g^2 + \mathcal{O}(g^4)) \\ \gamma_{mj}(g) &= g^{n_m} (B_{j1} + B_{j2}g^2 + \mathcal{O}(g^4)) \\ \gamma_j(g) &= g^{n_\gamma} (C_{j1} + C_{j2}g^2 + \mathcal{O}(g^4)) \end{aligned} \quad (77)$$

The relation between the schemes is supposed to be given by the perturbative equations

$$\begin{aligned} g_2 &= g_2(g_1) = g_1(1 + a_1g_1^2 + \mathcal{O}(g_1^4)), \\ m_2^2 &= m_1^2 z_m(g_1) = m_1^2(1 + b_1g_1^2 + \mathcal{O}(g_1^4)), \\ \phi_2 &= \sqrt{z}\phi_1 = \phi_1(1 + c_1g_1^2 + \mathcal{O}(g_1^4)), \end{aligned} \quad (78)$$

which induce the relations

$$g_1 = g_2(1 - a_1g_2^2 + \mathcal{O}(g_2^4)) \quad (79)$$

$$\begin{aligned} \beta_2(g_2) &= \mu \frac{\partial g_2}{\partial \mu} \\ &= \beta_1(g_1) \frac{\partial g_2}{\partial g_1} \\ &= \beta(g_1)(1 + 3a_1g_1^2 + \mathcal{O}(g_1^4)) \\ &= g_2^{n_\beta} [A_{11} + (A_{12} - n_\beta a_1 A_{11})g_2^2 + \mathcal{O}(g_2^4)] [1 + 3a_1g_2^2 + \mathcal{O}(g_2^4)] \\ &= g_2^{n_\beta} \{A_{11} + [A_{12} + (3 - n_\beta)a_1 A_{11}]g_2^2 + \mathcal{O}(g_2^4)\}, \end{aligned} \quad (80)$$

$$\begin{aligned} \gamma_{m2}(g_2) &= -\frac{\mu}{m_2^2} \frac{\partial m_2^2}{\partial \mu} \\ &= \gamma_{m1}(g_1) - \mu \frac{\partial \ln z_m(g_1)}{\partial \mu} \\ &= \gamma_{m1}(g_1) - \beta_1(g_1) \frac{\partial \ln z_m(g_1)}{\partial g_1} \\ &= g_2^{n_m} [B_{11} + (B_{12} - n_m a_1 B_{11})g_2^2 + \mathcal{O}(g_2^4)] \\ &\quad - g_2^{n_\beta} [A_{11} + (A_{12} - n_\beta a_1 A_{11})g_2^2 + \mathcal{O}(g_2^4)] 2b_1g_2(1 - a_1g_2^2 + \mathcal{O}(g_2^4)) \\ &= g_2^{n_m} [B_{j1} + (B_{j2} - n_m a_1 B_{j1})g_2^2 + \mathcal{O}(g_2^4)] - 2b_1g_2^{n_\beta+1} [A_{11} + (A_{12} - (n_\beta + 1)a_1 A_{11})g_2^2 + \mathcal{O}(g_2^4)] \end{aligned} \quad (81)$$

and

$$\begin{aligned} \gamma_2(g_2) &= \mu \frac{\partial \ln Z_2}{\partial \mu} \\ &= \mu \frac{\partial \ln Z_1}{\partial \mu} + \beta_1(g_1) \frac{\partial \ln z}{\partial g_1} \\ &= \gamma_{m1}(g_1) - \beta_1(g_1) \frac{\partial \ln z_m(g_1)}{\partial g_1} \\ &= g_2^{n_\gamma} [C_{11} + (C_{12} - n_\gamma a_1 C_{11})g_2^2 + \mathcal{O}(g_2^4)] \\ &\quad - g_2^{n_\beta} [A_{11} + (A_{12} - n_\beta a_1 A_{11})g_2^2 + \mathcal{O}(g_2^4)] 2c_1g_2(1 - a_1g_2^2 + \mathcal{O}(g_2^4)) \\ &= g_2^{n_\gamma} [B_{11} + (B_{12} - n_\gamma a_1 B_{11})g_2^2 + \mathcal{O}(g_2^4)] - 2c_1g_2^{n_\beta+1} [A_{11} + (A_{12} - (n_\beta + 1)a_1 A_{11})g_2^2 + \mathcal{O}(g_2^4)] \end{aligned} \quad (82)$$

for the RG coefficient functions. The first coefficient in the beta function is universal. When it starts with $\mathcal{O}(g^3)$ term then the second coefficient is universal, too. Universality holds for the first coefficient of γ_m and γ as long as they start not higher order than the beta function.

The bare and the renormalized coupling constants agree in the leading order of the perturbation expansion. Thus the bare coupling constants can be considered as the renormalized, running coupling constants taken at $\Lambda = \mu$ in some subtraction scheme. More precisely, one can artificially construct a subtraction scheme which reproduces the bare parameters of the theory as the running parameters considered at the cut-off in any given order of the perturbation expansion. The perturbative universality, spelled out above, applies to the bare and renormalized coefficient functions introduced in Eqs. (56), (58).

The question of the dependence on the choice of the regulator will be considered in section IV.

C. Dimensional transmutation

A renormalized trajectory of a theory with n coupling constant is a curve in n -dimensions. Its points correspond to the choice of parameters which reproduce the same physics and they can be parameterized in a natural manner by the cut-off. The fixing of the theory by the renormalization conditions is the selection of a renormalized trajectory. But the curves in n -dimensional space form an $n - 1$ -dimensional manifold. Thus the removal of the cut-off leads to the loss of a free, adjustable parameter in renormalizable theories.

This phenomenon is more spectacular in theories with a single massless coupling constant g . These theories have no free parameter whatsoever according to this argument but they still must contain an intrinsic dimensional parameters to allow some non-trivial cut-off dependence $g(\Lambda)$ for the dimensionless coupling constant. We can consider QCD as an example in the approximation where the quark masses are set to zero and the chiral symmetry breaking alone is responsible to the hadron masses. This theory is parameter free because its only parameter, the Yang-Mills coupling constant is defined on an arbitrary subtraction scale only. It is the combination of this scale and the corresponding value of the coupling constant which together fix the theory. This example suggests the presence of another interesting mechanism, too. The hadrons are massive in this model. What gives their scale if the only (not free) parameter is dimensionless? It is clearly a combination of the cut-off and the bare coupling constant in the bare perturbation expansion or the combination of the subtraction scale and the value of the running coupling constant in the renormalized perturbation expansion. This phenomenon is called dimensional transmutation.

Let us consider an observable of mass dimension, M , say a particle mass in a renormalizable theory with a single, dimensionless parameter g . Due to dimensional reason we have the form $M(g_B, \Lambda) = \Lambda f_B(g_B)$. The convergence when the cut-off is removed implies

$$0 = \Lambda \frac{dM}{d\Lambda} = M + \beta_B \frac{\partial M}{\partial g_B}. \quad (83)$$

When the renormalized theory is considered, defined at the subtraction scale μ then $M(g, \mu) = \mu f(g)$ and

$$0 = \mu \frac{dM}{d\mu} = M + \beta \frac{\partial M}{\partial g}. \quad (84)$$

The integration of these equations yields

$$M = \mu C e^{-\int \frac{dg}{\beta}} = \Lambda C_B e^{-\int \frac{dg_B}{\beta_B}}. \quad (85)$$

By assuming the leading order forms $\beta_B(g) = \beta_1 g^{n_\beta} (1 + \mathcal{O}(g^2))$, $\beta(g) = \beta_1 g^{n_\beta} (1 + \mathcal{O}(g^2))$ with $n_\beta \neq 1$ we find

$$M = \mu C e^{\frac{1}{(n_\beta - 1)\beta_1 g^{n_\beta - 1}(\mu)}} = \Lambda C_B e^{\frac{1}{(n_\beta - 1)\beta_1 g_B^{n_\beta - 1}(\Lambda)}}. \quad (86)$$

These equations show that the subtraction scale dependence of the running coupling constant and the cut-off dependence of the bare one must satisfy well defined relations in order to keep the physical content of the theory unique. Obviously, the coefficients C and C_B are non-perturbative in their origin. The first equation indicates that the perturbation expansion is reliable well above the characteristic energy scale M . The second equation renders perturbative renormalization unreliable for non-asymptotically free theories where $\beta_1 > 0$.

Another use of Eq. (85) is the introduction of a characteristic scale, the Λ -parameter of the theory, defined as the right hand side of Eq. (85) for $C = 1$ or $C_B = 1$ and some reasonable choice for the lower limit of integration in the

exponent. In the framework of the perturbation expansion, $\beta(g) = \mathcal{O}(g^{n_\beta})$, the simplest choice is $g = \infty$, leading to the definition

$$\Lambda_R = \mu e^{\int_g^\infty \frac{dg}{\beta}} \approx \mu e^{\frac{1}{(n_\beta-1)\beta_1 g^{n_\beta-1}(\mu)}} \quad (87)$$

for the given subtraction scheme. The leading order expression of the running coupling constant is then

$$g^{n_\beta-1}(\mu) = \frac{1}{-\frac{n_\beta-1}{2}\beta_1 \ln \frac{\mu^2}{\Lambda_R^2}}, \quad (88)$$

corresponding to writing the last equation in Eqs. (67) as

$$g^{n_\beta-1} = \frac{1}{\frac{1}{g^{n_\beta-1}} - \beta_1 \frac{n_\beta-1}{2} \ln \frac{\mu^2}{\mu'^2}} \quad (89)$$

and defining Λ_R by the equation

$$\frac{1}{g^{n_\beta-1}} = \beta_1 \frac{n_\beta-1}{2} \ln \frac{\mu'^2}{\Lambda_R^2}. \quad (90)$$

The second equation of Eqs. (86) allow the repetition of these steps for the bare theory. The Λ -parameter of the bare theory,

$$\Lambda_B = \Lambda e^{\int_{g_B}^\infty \frac{dg}{\beta_B}} \approx \Lambda e^{\frac{1}{(n_\beta-1)\beta_1 g_B^{n_\beta-1}(\Lambda)}}, \quad (91)$$

is a cut-off independent finite scale requiring the running

$$g_B^{n_\beta-1}(\Lambda) = \frac{1}{-\frac{n_\beta-1}{2}\beta_1 \ln \frac{\Lambda^2}{\Lambda_B^2}}, \quad (92)$$

of the bare coupling constant with the cut-off. Eq. (91) show the clearest manner the origin of dimensional transmutation: Imagine the calculation of a particle mass in the bare theory. Each term in the perturbation series is proportional to Λ . A partial resummation of the series, as carried out by the solution of the RG equation or in the saddle-point expansion, generates exponential functions of the bare coupling constant, just as the last factor in Eq. (91). These terms are just $\mathcal{O}(\Lambda^{-1})$ and the product remains finite whatever large values Λ takes. The emergence of these unusually small combination of the coupling constant prevents the cut-off to decouple.

The introduction of the Λ -parameters allows us to express the particle mass in Eqs. (85) simply as

$$M = C\Lambda_R = C_B\Lambda. \quad (93)$$

We can now see clearer the meaning of 'no free-parameter'. All observable becomes a simple number time an appropriate power of the Λ -parameter, in particular dimensionless ratios of observables are constant. The Λ -parameter sets the unit of scale and no other freedom is left.

As mentioned above, the renormalization conditions (33) can not always be inverted for the bare parameters. It may happen that for some value of the cut-off these equations yield diverging parameters and the cut-off can not be pushed forward anymore. It is clear from Eqs. (88) and (92) that the bare and the renormalized coupling constant have Landau-poles at Λ_R and Λ_B , respectively.

Let us now compare the Λ -parameter of two different subtraction schemes relating the running coupling constants as in the first equation of Eqs. (78). The leading order expression in Eqs. (87) for the Λ -parameters gives

$$\begin{aligned} \Lambda_2 &\approx \mu e^{\frac{1}{(n_\beta-1)\beta_1 g_2^{n_\beta-1}}} \\ &\approx \mu e^{\frac{1}{(n_\beta-1)\beta_1 g_1^{n_\beta-1} (1+a_1 g_1^2)^{n_\beta-1}}} \\ &\approx \mu e^{\frac{1-(n_\beta-1)a_1 g_1^2}{(n_\beta-1)\beta_1 g_1^{n_\beta-1}}} \\ &\approx \Lambda_1 e^{-\frac{a_1}{\beta_1 g_1^{n_\beta-3}}}. \end{aligned} \quad (94)$$

We note finally that the Λ -parameters can be introduced in theories with mass parameters. This parameter is simply the crossover scale, separating the UV and IR scaling regimes. In the multiplicative renormalization procedure one ignores terms of $\mathcal{O}(\Lambda^{-1})$ and arrives at the UV scaling laws. In another scheme where the contributions $\mathcal{O}(\Lambda)$ are ignored as $\Lambda \rightarrow 0$ one discovers the asymptotical IR scalings. These scaling regimes are separated by the intrinsic scale of the theory, the Λ -parameter. In a more formal manner, one can remove the dimension of the mass parameter with the cut-off and can look for the Λ -parameter of the resulting theory. This intrinsic scale, generated by the conspiracy of the higher order terms of the perturbation expansion and the cut-off, breaks the classical scaling laws, dictated by dimensional analysis and is the source of anomalous dimension.

D. Logs

The sub-divergences generate logarithmic corrections which may become dangerously at large for large enough energy. These logarithms can be resummed by integrating out the renormalization group equation. We shall see this in the case of the propagator where the higher-order contributions to the self-energy give

$$G^{(2)} \approx \frac{1}{p^2} \left[1 + \mathcal{O} \left(\ln \frac{p^2}{\mu^2} \right) \right] \quad (95)$$

and the logarithm may render the perturbation expansion invalid.

First we prove that the coefficient of the factor $1/p^2$ in the propagator is a polynomial of $\ln \frac{p^2}{\mu^2}$, more precisely,

$$G^{(2)} = \frac{1}{p^2} \sum_{n=0}^{\infty} g^{2n} P^{(n)} \left(\frac{p^2}{\mu^2} \right) \quad (96)$$

where $P^{(n)}(x)$ an n -th order polynomial of $\ln x$. We start with the 'renormalized' RG equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma \right] G^{(2)}(p; g, \mu) = 0 \quad (97)$$

where the mass is neglected. It gives

$$\frac{\partial}{\partial \ln \mu} \sum_{n=0}^{\infty} g^{2n} P^{(n)} \left(\frac{p^2}{\mu^2} \right) = - \left[\beta \frac{\partial}{\partial g} + \gamma \right] \sum_{n=0}^{\infty} g^{2n} P^{(n)} \left(\frac{p^2}{\mu^2} \right) \quad (98)$$

for the our ansatz. The integration leads to

$$\sum_{n=0}^{\infty} g^{2n} P^{(n)} \left(\frac{p^2}{\mu^2} \right) = \text{const} - \int_0^{\ln \mu} d \ln \mu' \left[\beta \frac{\partial}{\partial g} + \gamma \right] \sum_{n=0}^{\infty} g^{2n} P^{(n)} \left(\frac{p^2}{\mu'^2} \right) \quad (99)$$

We assume $n_\beta = n_\gamma + 1$ in Eqs. (77) when the RG equation can be solved iteratively because $\beta \frac{\partial}{\partial g} + \gamma$ increases the power of the coupling constant by n_γ . In the leading order we have $P^{(0)} = \text{const.}$ and $P^{(n)}$ for $n > 0$ will be a polynomial of order n in $\ln \mu$.

We now show that the integration of the one-loop RG equation resums the leading logarithms. For this end we introduce the index L for each contribution in the series as the half of the power of the coupling constant minus the power of the logarithm, $L = n - \ell$. The point is that all operator of the one-loop RG equation, $\mu \frac{\partial}{\partial \mu}$, $\beta \frac{\partial}{\partial g}$ and γ increases L by one and in particular terms with different L values are not mixed. Thus the leading logs, the sum of terms with $L = 0$ is a solution of the one-loop RG equation. It is easy to see in a similar manner that integration of the k -loop RG equation resums the leading logs together with their $k - 1$ sub-leading terms.

The resummation of the potentially dangerous logarithms of the perturbation expansion is an important method to connect observables with well separated scales. For example, when the asymptotically high energy limit of Green functions is sought the direct perturbative calculation picks up the logarithms as in Eq. (95) and renders the perturbation expansion unusable. But by the integration of the RG equation in arriving at Eq. (74) one resums the dangerous logs and the result will again be perturbative when the expansion is reorganized in the properly chosen running coupling constant.

The base point of the traditional perturbation expansion is the free system. The partial resummation of the perturbation expansion, performed by the renormalization group, consists of reorganizing the expansion around

'quasi-particles', the already interacting degrees of freedom. One gains if the interaction is weaker among the quasi-particles than among the original, bare particles. These two strategies can be demonstrated by integrating the differential equation

$$\frac{dx(t)}{dt} = F(x(t)). \quad (100)$$

The naive perturbation expansion is similar to the expansion of the right hand side around $x = 0$,

$$x(t + \Delta t) = x(t) + \Delta t \sum_n f_n x^n + \mathcal{O}(\Delta t^2). \quad (101)$$

The improved scheme is the expansion around the running $x(t)$,

$$x(t + \Delta t) = x(t) + \Delta t F(x(t)) + \mathcal{O}(\Delta t^2), \quad (102)$$

where the higher order terms of this expansion can be neglected.

III. EFFECTIVE THEORIES

The renormalizable models, discussed so far, represent very special theories. In fact, renormalizability, the convergence of the physical content when the cut-off is removed assumes that the underlying dynamics is fully, down to the smallest microscopic details is reproduced by the elementary interactions displayed in the interaction part of the Lagrangian. We, finite being, shall never have such a complete and satisfactory picture of physics. There will always be the possibility of hitherto unknown, heavy particles which mediate some interactions not yet discovered in our observations of limited resolution.

To be realistic, we have to be ready to work with non-renormalizable models, to face the problem of determining some of the infinitely many parameters and to try to extract some approximative predictions with controllable accuracy. The decoupling theorem, presented first below, is useful to characterize the effective interaction vertices arising in the low energy dynamics due to existence of some virtual, high energy particles. The strategy to use the resulting non-renormalizable theory to extract predictions is discussed next.

A. Decoupling of heavy particles

Let us imagine that we send a particle mass M to infinity in a renormalizable theory. How does the low energy sector of this model look like below the energy scale M ? The renormalizable coupling constants parameterize the possible finite energy dynamics. The heavy particle leaves virtual, suppressed trace in the low energy effective theory, obtained after the elimination of the heavy particle. Thus one expects that the heavy particles induce small effective vertices beyond the renormalizable parameters of the effective theory. The coupling constants of the effective theory can be regrouped according to two different strategies. One is the separation of the renormalizable parameters from the non-renormalizable one. Another scheme is to identify the coupling constants which receive large contributions from the heavy particles from those which are suppressed by a power of the heavy particle mass. The decoupling theorem, valid for renormalizable theories, states that these two classification schemes are equivalent.

Let us start with an example, with a theory with two particles in $d = 6$, the Lagrangian being

$$L_R = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{M^2}{2}\phi^2 - \mu^{3-\frac{d}{2}} \left(\frac{g}{3!}\phi^3 + \frac{G}{2}\phi\Phi^2 \right) - \mu^{\frac{d}{2}-3}h\phi \quad (103)$$

and we seek the low energy effective theory in the limit $M/m \rightarrow \infty$. For this end we need the power-counting scheme for the 1PI vertices in terms of the heavy mass M .

Let us start with a finite graph, shown in Fig. 7 (a) for vanishing external momenta,

$$\Gamma^{(2)} = G^4 \int_k \frac{1}{(k^2 - M^2)^4} = \frac{G^4}{384\pi^3 M^2}. \quad (104)$$

Being finite, it scales according to the classical dimensions, M^{-2} , ie. superficial degree of the loop-integral. This is because for each momentum square in the propagator there is a mass square, too. But may happen that another dimensional parameter, such as the light mass occur in the integrand and the finite integral follows different power in

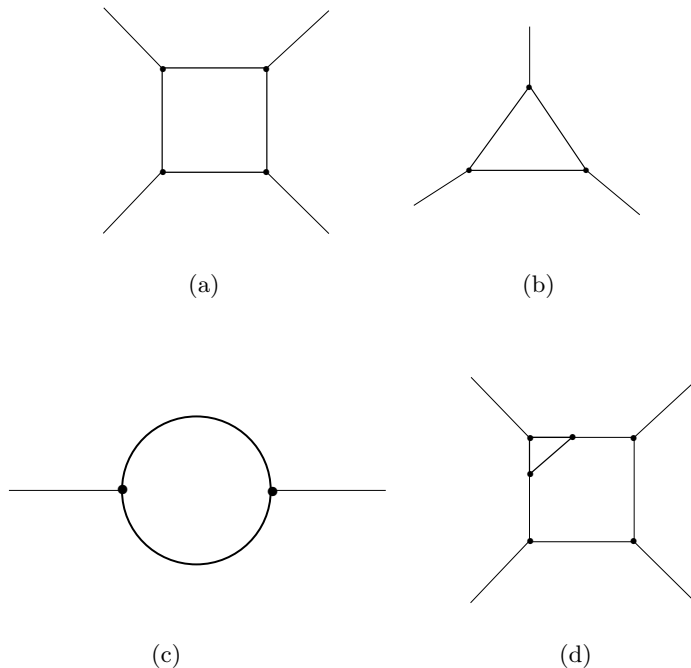


FIG. 7: Effective vertices for the light particle.

the heavy mass than the superficial degree of divergence. What remains valid is that all UV finite graphs containing heavy lines are suppressed as $M \rightarrow \infty$. We can expand in the external momenta when $M \rightarrow \infty$ and these graphs generate local, suppressed vertices in the effective theory,

$$\Delta g_4 = \frac{G^4}{384\pi^3 M^2} \quad (105)$$

in this particular case.

We introduce the degree of divergence in the heavy mass, $\omega_M[I]$ for the loop-integrals which is negative whenever the primitive degree of divergence, $\omega[I]$ is negative, too.

The logarithmically divergent effective vertex of Fig. 7 (b) is

$$\begin{aligned} \Gamma_R^{(3)} &= \frac{G^3}{64\pi^3} \left[\frac{\gamma}{2} + \int_0^1 dx \int_0^{1-x} dy \ln \frac{M^2 + (p_1^2 x + p_2^2 y)(1-x-y) + p_3 xy}{4\pi\mu^2} \right] \\ &= \frac{G^3}{128\pi^3} \left[\gamma + \ln \frac{M^2}{4\pi\mu^2} + \mathcal{O}\left(\frac{p_i^2}{M^2}\right) \right] \end{aligned} \quad (106)$$

in minimal subtraction. We find a logarithmic divergence in the heavy mass, $\omega_M[I] = 0 = \omega[I]$, and the external momentum dependence is again polynomial, eg. local. Such a local nature of the effective vertices can be understood by recalling that the heavy particle is strongly off-shell and can not propagator far at low energies. The effective vertex

$$\Delta g_3 = -\frac{G^3}{128\pi^3} \left[\gamma + \ln \frac{M^2}{4\pi\mu^2} \right] \quad (107)$$

is non-vanishing in the heavy mass limit and the divergence type is the same as in the cut-off.

A stronger, quadratic divergence is observed in the effective vertex of Fig. 7 (c),

$$\begin{aligned} \Gamma_R^{(4)} &= \frac{G^2}{128\pi^3} \left[(\gamma - 1) \left(M^2 - \frac{p^2}{6} \right) + \int_0^1 dx [M^2 + p^2 x(1-x)] \ln \frac{M^2 + p^2 x(1-x)}{4\pi\mu^2} \right] \\ &= \frac{G^2}{128\pi^3} \left[M^2 \left(\gamma - 1 + \ln \frac{M^2}{4\pi\mu^2} \right) + \frac{p^2}{6} \left(\gamma + \ln \frac{M^2}{4\pi\mu^2} \right) \right] + \mathcal{O}\left(\frac{p^2}{M^2}\right) \end{aligned} \quad (108)$$

in minimal subtraction again. The relation $\omega_M[I] = \omega[I]$ is confirmed in this case, too. The resulting effective vertices are

$$\begin{aligned}\Delta Z &= -\frac{G^2}{768\pi^3} \left(\gamma + \ln \frac{M^2}{4\pi\mu^2} \right) \\ \Delta m^2 &= \frac{G^2}{128\pi^3} M^2 \left(\gamma - 1 + \ln \frac{M^2}{4\pi\mu^2} \right).\end{aligned}\tag{109}$$

In general, the relation $\omega_M[I] = \omega[I]$ is preserved for graphs with primitive divergences. In fact, after deriving the graph few times with respect to the external momenta it becomes UV finite and suppressed as $M \rightarrow \infty$. But the UV divergent terms, eliminated by the BPHZ procedure, contain the contribution of the loop-integral from arbitrary large internal loop variable k , in particular from the region $k^2 \approx M^2$. The contribution of this region is $M^{\omega[I]}$ for $\omega[I] \neq 0$ or $\ln M$ for $\omega[I] = 0$.

When the UV divergence comes from a sub-graph containing heavy and light particles, as in Fig. 7 (d), then the generated effective vertex is the contact term obtained by pulling together the heavy lines into a single point. The relation $\omega_M[I] = \omega[I]$ is preserved because this is a primitive divergence of the sub-graph where each internal line is heavy. This relation assures that $\omega_M[I] = \omega[I]$ continues to be valid for UV divergent graphs even in the presence of properly subtracted sub-divergences.

The summary of the situation is the following:

1. A loop-integral containing heavy particle propagators can be non-vanishing in the limit $M \rightarrow \infty$ only if the loop-momentum k runs un-suppressed at scales $k^2 \approx M^2$.
 - The UV finite graphs give suppressed effective vertices.
 - The UV divergent graphs are not suppressed by the heavy mass.
2. According to Weinberg theorem the power counting correctly predicts the scaling of the graphs when some external momenta diverges up to logarithmic corrections. The application of such a power counting for the overall or sub-divergences of the graphs establishes $\omega_M[I] = \omega[I]$ for $\omega[I] \geq 0$ because $k^2 \approx M^2$ for the important regions.
3. The dimension of the effective vertex is the sum of the dimensions of the coupling constants and the loop-integral, $[\text{vertex}] = [g] + [I]$. Owing to $[g] \geq 0$ we have $[\text{vertex}] \geq [I]$. Thus each non-suppressed effective vertex in the limit $M \rightarrow \infty$ has non-negative dimension and thereby qualifies as renormalizable.
4. The effective vertices, obtained by expanding in the external momenta are always local.

There are subtraction schemes with manifest decoupling, they contain all un-suppressed effective vertices after the simple elimination of the heavy particle. The mass-independent version of BPHZ is such a scheme because the counterterms, corresponding to the UV divergent terms of the expansion in the external momenta contain just the un-suppressed effective vertices. Non-manifest decoupling schemes can be obtained from this by a finite renormalization without changing the divergence structure.

The decoupling theorem shed more light on the question of when and why can we be satisfied with renormalizable theories. A necessary condition for the dominance of the renormalizable part of the effective theory is the strong scale separation between the heavy mass and the observation scales, when the non-renormalizable effective coupling constants are strongly suppressed. But this is not sufficient, sensitive enough measurement can still uncover the virtual vacuum polarizations if the quantum channel in question has no contribution from the un-suppressed, renormalizable part. This explains the possibility of seeing weak interaction effects well below 100GeV.

The decoupling theorem is not always applicable when there is a heavy particle. We used in the power counting argument that the heavy particle scale comes a large quantity of dimension energy. It may happen that the mass, generated by spontaneous symmetry breaking becomes large be a large dimensionless coupling. This kind of heavy particle non-decoupling. The heavy but stable particles may play important role at low energies, well below their mass scale as witnessed by non-relativistic physics, in general. This is another exception of the decoupling theorem where the symmetry which reinforces the stability of the heavy particles keeps them in the same time on mass-shell in an anti-particle poor environment.

B. Effective theories

The small but eventually important non-renormalizable coupling constants, arising from the presence of some unknown, heavy particles are unavoidable ingredients of our models. These vertices are non-renormalizable and are

excluded from consideration in the 'orthodox' approach to quantum field theory on the ground that non-renormalizable theories have no predictive power due to the need of infinitely many counterterms and renormalization conditions. But one can save some of the predictive power. In fact, the non-renormalizable coupling constants have negative energy dimension and seem to be less important at low energy. Thus for any desired accuracy of the results we can truncate the theory and retain the couplings with not too negative energy dimensions whose number is limited.

To make this argument more precise we have to introduce two scales to specify the separation of the the original, supposedly renormalizable and the low energy effective theory. Let M denote a heavy particle scale. The other scale $\Lambda_{\text{eff}} \ll M$, the cut-off for the low energy effective theory. We set $\Lambda_{\text{eff}} \ll M$ because the effective theory well below the heavy mass threshold contains local effective vertices only. A typical example is the Weinberg-Salam model as the original theory and the Fermi contact interaction as the effective theory with the separation scale $M \approx 100\text{GeV}$. The troublesome, non-renormalizable vertices of the low energy effective theory are suppressed by the heavy mass scale and are of the form gM^{-n} , such as the Fermi coupling, $\frac{g}{M^2}(\bar{\psi}\psi)^2$. The one-loop self energy generated by this vertex,

$$\Sigma = \frac{g}{M^2} \int_k \frac{m}{k^2 - m^2} \quad (110)$$

where m denotes the light fermion mass is quadratically divergent. If we choose a cut-off in the effective theory in the vicinity of the separation scale M then we find $\Sigma = \mathcal{O}(M^0)$ and the microscopical details reappear in the low energy dynamics. This should be avoided because the effective interaction vertices start to be non-local at this scale due to the retardation effects of the heavy particle dynamics. As soon as we set $\Lambda_{\text{eff}} \ll M$ we recover $\Sigma \approx \Lambda_{\text{eff}}^2/M^2$ with an expected suppression factor.

The generic suppression of the contribution of the non-renormalizable vertices is installed by a sufficiently low cut-off in the effective theory. This suppression mechanism will be the central issue in constructing the Wegner-Houghton equation, too.

It is important to realize that the expectation values do not depend on the choice of the scale, Λ_{eff} , separating the original and effective theory in the exact solution of the problem. In fact, the dependence of the effective theory on the choice of the value of the cut-off is exactly compensated by the cut-off dependence of the coupling constants of the effective theory, Λ_{eff} being a book-keeping parameter only, to govern the separation of the original theory into high and low energy sectors. When truncation is made then Λ_{eff} -dependence appears. The minimization of this dependence can be used to optimize the approximation scheme.

To present the general power counting of the suppression of the contributions of the non-renormalizable vertices we consider the effective Lagrangian for a scalar field,

$$L = f^4 \sum_n \frac{g_n}{M^{d_n}} \mathcal{O}_n(\tilde{\phi}) \quad (111)$$

where f , M are mass scales, g_n denote dimensionless couplings and the dimension of the field variable $\tilde{\phi} = \phi/K$ is removed by the scale K . The local expression $\mathcal{O}_n(\tilde{\phi})$ contain d_n space-time derivatives acting on the field variable. Let us estimate the magnitude of a graph with I internal and E external lines, $V_{\ell,d}$ vertices with ℓ lines attached and d space-time derivatives acting on them.

We shall need the identities,

$$\begin{aligned} 2I + E &= \sum_{\ell,d} \ell V_{\ell,d}, \\ L &= I - 1 + \sum_{\ell,d} V_{\ell,d}. \end{aligned} \quad (112)$$

The first counts the total number of lines attached to the vertices on two different manners, the second give the number of loops in the graph, the number of independent four-momenta.

Let us start with the integrand of loop-integral,

$$\underbrace{\prod_{i=1}^I \left[\int \frac{d^4 p_i}{(2\pi)^4} \frac{M^2 K^2}{f^4} \frac{1}{p_i^2 - m^2} \right]}_{\text{internal lines}} \cdot \underbrace{\prod_{\ell,d} \left[(2\pi)^4 \delta^{(4)}(p) \left(\frac{p}{M} \right)^d \frac{f^4}{K^\ell} \right]^{V_{\ell,d}}}_{\text{vertices}}, \quad (113)$$

and estimate the loop-integration for values of the external energy \mathcal{E} which are comparable or larger than the light mass, m . According to Weinberg theorem we can use power counting and estimate the result by dimensional argument. No observable depends on the cut-off of the effective theory, Λ_{eff} , when cancellations among different graphs are taken

into account therefore this scale can be ignored. Another way to see this is to use dimensional regularization and minimal subtraction where the scale of the loop-integrals is given by the external momenta or the mass. Thus the power counting leads to the replacement $I \rightarrow \mathcal{E}^I$ to find the leading contribution in m/M or \mathcal{E}/M ,

$$\begin{aligned}
G &\approx \mathcal{E}^{2I-4-4\sum_{\ell,d} V_{\ell,d}} \left(\frac{M^2 K^2}{f^4}\right)^I \left(\frac{\mathcal{E}}{M}\right)^{\sum_{\ell,d} d V_{\ell,d}} \left(\frac{f^4}{K^\ell}\right)^{\sum_{\ell,d} V_{\ell,d}} \\
&= f^4 K^{-E} \left(\frac{M^2}{f^2}\right)^{2L} \left(\frac{\mathcal{E}^2}{M^2}\right)^{I+\frac{1}{2}\sum_{\ell,d} d V_{\ell,d}} \\
&= f^4 K^{-E} \left(\frac{M\mathcal{E}}{f^2}\right)^{2L} \left(\frac{\mathcal{E}^2}{M^2}\right)^{1+\sum_{\ell,d} (\frac{d}{2}-1)V_{\ell,d}}
\end{aligned} \tag{114}$$

after factorizing off the energy-momentum conserving Dirac-delta. The lesson of the second equation is that the power of \mathcal{E}/M is always positive, each internal line and derivative at the vertex contributes to the suppression. Furthermore, the perturbation expansion in the effective theory requires $\mathcal{E}^2 \ll M^2 \ll f^2$ but the value of f remains arbitrary.

The second equation is useful for the effective theory for Goldstone bosons because these particles have gradient coupling and vertices with $d \geq \ell$ appear only. Let us consider for the sake of an example the $U(1)$ symmetric scalar model described by the Lagrangian

$$L = \partial^\mu \phi^* \partial_\mu \phi + m^2 \phi^* \phi - \frac{g}{4} (\phi^* \phi)^2. \tag{115}$$

The parameterization $\phi = \rho e^{i\theta}$ gives the Lagrangian

$$L = \rho^2 (\partial\theta)^2 + (\partial\rho)^2 + m^2 \rho^2 - \frac{g}{2} \rho^4. \tag{116}$$

The mass square of the ρ particles is $m_H^2 = 4m^2$, their field fluctuates around $\rho_0 = \sqrt{m^2/g}$ and the effective Lagrangian for the Goldstone modes is of the form (111) with $f^4 = \rho_0^2 m_H^2$, $M^2 = m_H^2$ and $K = 1$. The ratio $M^4/f^4 = 4g$ can remain arbitrary as long as $\mathcal{E} \ll \min(M, f^2/M)$ according to the last equation in (114). In other words, the low energy dynamics of the Goldstone modes is perturbative even if the original theory is strongly coupled.

Heavy but stable particles lead to a different kind of low energy effective theory. This theory can be constructed by eliminating the anti-particles only, leaving the particles intact. The rest mass energy of the particles couples only to gravity and will be neglected. We recover a non-relativistic theory in this manner where the velocity p/m , counted in units of the speed of light, plays the role of the small parameter.

The cut-off independence of the effective theories, remarked above, shows the characteristic feature of renormalizable theories, namely that the given interaction captured by the renormalizable vertices can be extrapolated to arbitrary high energies, from a different angle. The cut-off independence of the effective theories is achieved in a true, non-renormalizable effective theory by the presence of infinitely many effective vertices. When the cut-off is increased and the calculation is pursued to higher energies then more effective coupling constants and higher order of the perturbation expansion is needed for a given level of accuracy. A natural renormalizable theory contains such vertices which allow the increase of energy without losing accuracy, up to logarithmic corrections. Or, in the language of the decoupling theorem, the specialty of the renormalizable low energy theories with manifest decoupling is that they are accurate in order $\mathcal{O}(M^0)$ already. Non-manifest decoupling which leaves infinitely many small non-renormalizable vertices behind requires their use and renders the high energy extrapolation more cumbersome.

IV. HIGH ENERGY VS. STATISTICAL PHYSICS

The renormalization group appears in different manners in High Energy and Statistical Physics. The most obvious difference is the direction of the evolution. One looks for long distance effects in Statistical Physics and drive the RG towards the IR direction. In High Energy Physics one is usually interested in the removal of UV divergences and in high energy phenomena. Thus the RG equations are solved to approach the UV regime. Another difference which is more a convention rather than essence is that one tends to use running coupling constants, quantities read off from expectation values, to construct the scale dependence in Quantum Field Theory while the lattice Hamiltonian with its bare quantities what usually occurs in Statistical Physics. Finally, a technical difference is that we try to remove the cut-off in Quantum Field Theory but it is kept at a physical value in Solid State Physics.

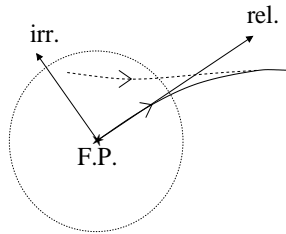


FIG. 8: The renormalized trajectory in the vicinity of an UV fixed point F.P. in the presence of a relevant and an irrelevant coupling whose axes are shown. The solid and dotted lines represent the trajectory of models with a single relevant and a relevant plus and an irrelevant coupling. The circle indicates the validity of asymptotic scaling obtained by the linearization of the scaling laws around the fixed-point.

A. Critical phenomena and renormalizability

Let us look into these difference more closely by considering a lattice gas description of electrons in a conducting band of a solid in the second quantized, Quantum Field Theoretical framework. There are three important length scales even in the simplest case. The shortest is the lattice spacing a , the minimal distance, and it restricts the momenta into the interval $0 < p < \Lambda$ with $\Lambda = 2\pi/a$. There is an internal length scale in the dynamics, the correlation length ξ , its value measured in lattice spacing units is $\xi_L = \xi/a \gg 1$. One is usually interested in transport processes in solid which take place at distances $x \gtrsim \xi$.

When a high energy physics problem is considered one uses relativistic quantum field theory which has analogous scales. To avoid confusion we set $\hbar = c = 1$ and use length as the dimension of scale in this discussion. The shortest length scale is the cut-off, $a = 2\pi/\Lambda$. What is the analogy of the correlation length? It is well known that one can not localize a particle within the distance much shorter than its Compton wave-length, $\lambda_C = 1/m$ because such a localization requires momenta $p \gg 1/\lambda_C = m$ and pair-creation would smear out the position of the particle. In other words, the quantum field is strongly correlated within the distance λ_C which plays the role of correlation length, $\xi = 1/m$. Finally, there is the length scale of the observation, $x = 1/p \gtrsim 1/m$.

The similarity of the two problems can be made even more explicit when the high energy physics problem is present in the framework of a lattice regulated quantum field theory. The best is to imagine such a regulator in this section.

Suppose that we tune the environmental parameter to drive our solid through a critical point, a genuine long range phenomenon. The ratio of the correlation length ξ and the fixed cut-off a diverges at the critical point, $\xi_L \rightarrow \infty$ but the observational scale is supposed to be still longer than ξ . The singularities of susceptibilities etc. build up due to the many order of magnitude non-trivial dynamics within the scale interval $a < x < \xi$ and they are resummed by finding the IR end of the renormalized trajectory, the running of the bare parameters of the lattice Hamiltonian.

The renormalization of a relativistic quantum field theoretical model consists of sending the cut-off Λ to infinity by keeping the physics, in particular the particle mass m fixed. Thus the ratio of the fixed correlation length and the cutoff, $\lambda_C/a = \Lambda/2\pi m$ diverges. The dynamics spanning in the length scales $2\pi/\Lambda < x < 1/m$ pile up contributions which make up the UV divergence structure of the theory.

The first important lesson is that both the critical phenomena and the renormalization of a quantum field theory correspond to a fixed-point. The difference between the two cases is the direction we move on the renormalized trajectory and the choice of physical units in which either the cut-off or the correlation length is fixed.

The critical system is an UV fixed point in Statistical Physics and it provides us with a classification scheme of local operators. Suppose that we have a relevant and an irrelevant coupling constant in our model whose renormalized trajectory is sketched in Fig. 8. We follow the trajectory towards the IR direction, along the arrow in describing the observables in critical system with $\xi_L \rightarrow \infty$. When the cut-off is removed, $\Lambda \rightarrow \infty$ or the lattice spacing of the lattice regulated quantum field theory tends to zero, $a \rightarrow 0$ the correlation length diverges in units of the cut-off, $\lambda_C/a = \Lambda/2\pi m \rightarrow \infty$ and we move on the trajectory towards the UV direction. The trajectory with relevant coupling only leads us to the fixed-point but whenever there is a non-vanishing irrelevant coupling then the trajectory turns away from the fixed-point in the limit $\Lambda \rightarrow \infty$. Thus the system can be fine-tuned to a fixed-point and the cut-off can be removed in a convergent manner in the presence of relevant or marginal coupling constants only but irrelevant operators send the large Λ system to different type of actions. The conclusion is that the relevant and marginal operators are renormalizable and the irrelevant operators are non-renormalizable. These remarks are summarized in Table I.

The truly marginality operators are hard to find. It may happen that an operator appears marginal up to some order of the perturbation expansion but this scale independence is usually broken by higher order contributions. For

Statistical Physics	Quantum Field Theory ($\hbar = c = 1$)
lattice spacing, minimal distance: a	cutoff: $\Lambda = \frac{2\pi}{a}$
correlation length: ξ	Compton wavelength: $\frac{1}{m}$
critical phenomenon: a is fixed, $\frac{\xi}{a} \rightarrow \infty$	renormalization: m is fixed, $\frac{\Lambda}{m} = \frac{2\pi}{\frac{1}{\xi}} = 2\pi\frac{\xi}{a} \rightarrow \infty$
UV fixed point	renormalized theory
relevant or marginal operator	renormalizable operator
irrelevant operator	non-renormalizable operator
universality	renormalizable theories cover all possible dynamics

TABLE I: Comparison of critical systems and the removal of the cut-off in relativistic quantum field theory.

example the truly renormalizable coupling constants of the power counting argument are dimensionless and appear marginal in the tree-level scaling relations. But already one-loop contribution to the beta-functions usually render them relevant, such in $d = 4$ $SU(N_c)$ Yang-Mills theories with N_f fermions,

$$\frac{g^2(\mu)}{(4\pi)^2} = \frac{\frac{g^2(\mu_0)}{(4\pi)^2}}{1 + \frac{g^2(\mu_0)}{(4\pi)^2} \frac{11N_c - 2N_f}{3} \ln \frac{\mu^2}{\mu_0^2}} \quad (117)$$

or irrelevant as in QED

$$\frac{e^2(\mu)}{4\pi} = \frac{\frac{e_0^2}{4\pi}}{1 + \frac{e_0^2}{4\pi} \frac{1}{3\pi} \ln \frac{\mu^2}{m^2}} \quad (118)$$

($\mu_0 = m$).

The correspondences of Table I represent non-perturbative results. But let us suppose now that the perturbation or loop-expansion is applicable to our system. Then the critical exponents will be obtained as a power series in the small parameter and their sign are decided by the first non-vanishing order. Thus it is reasonable to expect that operators which are relevant or irrelevant in the leading order remain the same when higher orders are included. The marginal operators may change into relevant or irrelevant, depending on the sign of the first non-vanishing contribution. The leading order classification comes from tree-level dimensional analysis, coupling constants with dimension of length appear small in the IR and are irrelevant. This translates the condition of non-renormalizability into $[g] < 0$ when we count the dimension in energy units. In a similar manner, couplings with negative length dimension are relevant and the condition $[g] > 0$ indeed belongs to super renormalizable coupling constants. The marginal couplings in Statistical Physics are the true renormalizable couplings of quantum field theory. This simple argument is equivalent with the replaces the BPHZ proof of renormalizability. Naturally the latter gives much more details but on the expense of transparency and clarity.

Finally, the universality stands for the disappearance of the irrelevant coupling constants from certain quantities, such as critical exponents, characterizing critical phenomena. In a similar manner, the finite energy content of a renormalized quantum field theory is parameterized by its renormalizable coupling constants.

Notice that relevance/irrelevance is not the best choice of words because they do not correspond to importance/unimportance. In fact, the irrelevant coupling constants play important and well measurable role in the dynamics at a finite energy, it is enough to recall the Fermi contact interaction or any multiple scattering process in nuclear or particle physics. These effective coupling constants, determined by some value of 1PI vertex functions might be small but still have important, finite effects. What universality claims is that all of the infinitely many irrelevant constant can be parameterized or expressed in terms of few relevant or marginal parameters in a renormalizable theory.

The renormalization group is actually a semigroup because the Kadanoff-Wilson blocking is applicable without limitation towards the IR direction only. The inverse motion, towards the UV is needed for the renormalization of the quantum field theoretical models. The renormalization semigroup becomes group for renormalizable models. When the inverse ceases to exist then we tumbled upon an UV Landau-pole. The always well defined nature of the Kadanoff-Wilson blocking assures that there is no true IR Landau-pole. Whenever we seem to find one then it must be made up by our approximations only, eg. the divergence of the one-loop running coupling constant in QCD is the signal that the dynamics can not be reproduced below Λ_{QCD} in terms of quarks and gluons. But the coupling strength corresponding to the true quasiparticles, hadrons, must remain regular in this regime.

Non-asymptotically free models may become strongly coupled at short distances and their renormalizability is beyond perturbation expansion. The numerical simulations of lattice regulated quantum field theories suggest that

asymptotically free models are renormalizable only. The U(1)- and the Higgs-sector of the Standard Model is non asymptotically free and assures us of the new physics waiting us beyond the intrinsic scale of the model.

B. Irrelevant operators

Let us have a closer look on the so far passive participants, the irrelevant coupling constants. Do they really drop out from the finite energy dynamics when the cut-off is removed? To find the answer we consider a model in two versions which differ in the values of some irrelevant coupling constants g_{irr} only, cf. Fig. 9. Let us start with version (1) and make blockings which increases the lattice spacing, $a \rightarrow sa$ and reduces the momentum-space cutoff $\Lambda \rightarrow \Lambda/s$ by a given factor s . The result is shown by the sequence of points on the renormalized trajectory. Let us now turn to version (2) and do the same. The two trajectories converge if the UV scaling regime is sufficiently long what is now assumed for simplicity. But this does not mean that the points on the two trajectories approach each others because they are usually 'out of phase' and the two sets remain displaced despite the overlap of the trajectory. Let us fine-tune the irrelevant coupling strengths in such manner that the points really approach each other and denote by $n^{(j)}$ the blocking step needed to make a point on the trajectory of version j approximately coincide with a point of the other trajectory. If the points agree then so do the two theories at that point for quantities expressed in units of the lattice spacing, such as the correlation length,

$$\frac{\xi_L(g_r, g_{irr}^{(1)})}{s^{n^{(1)}}} = \frac{\xi_L(g_r, g_{irr}^{(2)})}{s^{n^{(2)}}} \quad (119)$$

This equation, written for the correlation length in physical units reads

$$\xi(g_r, g_{irr}^{(2)}) = s^{n^{(2)} - n^{(1)}} \frac{a^{(2)}}{a^{(1)}} \xi(g_r, g_{irr}^{(1)}) \quad (120)$$

where $a^{(j)}$ denotes the initial lattice spacing for version j .

There are two ways to read the second equation:

1. The two versions of the theory are equivalent. Then $\xi(g_r, g_{irr}^{(2)}) = \xi(g_r, g_{irr}^{(1)})$ and $a^{(2)} = a^{(1)} s^{n^{(1)} - n^{(2)}} \neq a^{(1)}$ and the two versions must have different value of the cutoff. The non-trivial ratio of the cut-offs and the different number of blocking steps needed to bring the two versions to overlap compensate each other. The irrelevant couplings influence the actual value of the cut-off which reproduces the desired dynamics with the given relevant coupling constants.
2. The two cut-offs are equivalent, $a^{(2)} = a^{(1)}$: The dimensionless ratios are universal, $O(g_r, g_{irr}^{(2)}) = s^{d_O(n^{(2)} - n^{(1)})} O(g_r, g_{irr}^{(1)})$ in theories which share the same relevant coupling constants. The irrelevant parameters influence only a common scale in these family of theories.

We see again that irrelevance is not unimportance. In the language of Statistical Physics $g - g^* \rightarrow 0$ but g^* might be large.

Point 1 above, stating that the change of the irrelevant coupling constants induces a change in the cut-off, suggests a relation between these two ingredients of the theory. Actually, the regulator can be viewed as a special set of irrelevant operator in the theory, eg. the free propagator $G_0(p^2) = 1/p^2 - m^2 + (p^2)^n/\Lambda^{2n-2}$ with some $n > 0$ produces UV finite dynamics for a scalar particle because the contributions of states with $p \gg \Lambda$ are suppressed. Thus we can never get rid of irrelevant operators. But these irrelevant operators come with suppressed coupling strengths which just suppresses the would be UV divergences in the power counting and make the limit $\Lambda \rightarrow \infty$ convergent. Furthermore, the changing of the regulator or the subtraction scheme amounts to the modification of the irrelevant coupling constants and a common scale factor. This is a qualitative generalization of the argument presented at Eq. (94), that the change of the renormalization scheme induces a modification of the intrinsic scale to regulator dependence.

V. WHEN THE POWER COUNTING IS MISTAKEN: QUANTUM ANOMALIES

We have, up to now, emphasized the similarities of the renormalization group strategy in Statistical and High Energy Physics. But there is an obvious, fundamental difference, namely the UV divergence structure seems to be

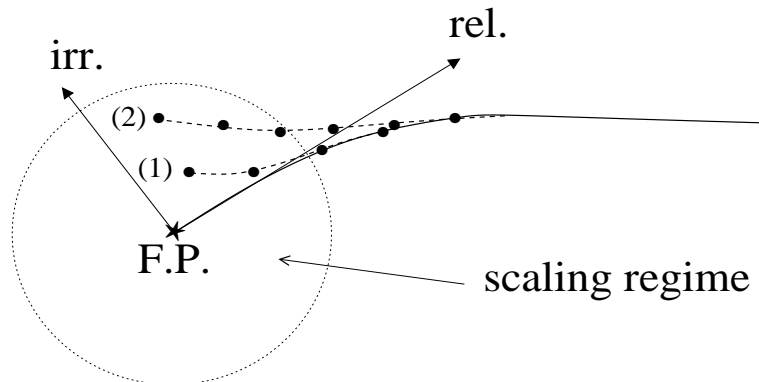


FIG. 9: The renormalized trajectory in the vicinity of an UV fixed point F.P. as in Fig. 8 for two different set of irrelevant coupling constants.

inextractably wound up with the renormalizable parameters in High Energy Physics but no UV divergences arise in Statistical Physics.

What binds UV divergences and relevance together? It is the power counting because both tree-level relevance and perturbative renormalizability is expressed by the same condition $[g] \geq 0$. On the one hand, it is obvious that quantities with positive energy dimensions assume large values in low energy units. On the other hand, the renormalized vertex $g\mathcal{O}$ induces the counterterm $\delta g\mathcal{O}$ where δg is a loop-integral with $[\delta g] = [g]$. Thus the primitive divergence of the counterterm is indeed equivalent with having non-negative dimension for the coupling constant.

What may come as a surprise is that the power counting correctly describes the necessary but not the sufficient conditions for the UV divergences. The scheme may fail to realize that a loop-integral with non-negative dimension is actually finite due to some cancellation. When this happens then the relevance and the UV divergences divorce. This kind of loop-integrals show a number of remarkable and confusingly different features which are hard to relate to each others. This might be the historical reason that these graphs went down with the name 'anomaly'.

Let us start with the remark that the particles propagate easily along fractal trajectories in Quantum Mechanics as opposed to Classical Mechanics where the trajectories are analytic. This can easily be seen by inspecting the path integral

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int dx_j e^{\frac{i}{\hbar} \Delta t \sum_{\ell=1}^N [\frac{m}{2} (\frac{x_\ell - x_{\ell-1}}{\Delta t})^2 - U(x_\ell)]} \quad (121)$$

corresponding to the system with Hamiltonian

$$H = \frac{p^2}{2m} + U(x). \quad (122)$$

The typical trajectories should have $\Delta x = x_\ell - x_{\ell-1} \approx \sqrt{\hbar \Delta t / m}$ because larger Δx leads to destructive interference and trajectories with smaller Δx occupy too small phase space. When the end-point dependence is negligible say for periodic trajectories then we can make the shift $x_\ell \rightarrow x_\ell - x_{\ell-1}$ in the integral variables and find $\langle \Delta x^2 \rangle = -i\hbar \Delta t / m$, $\langle v^2 \rangle = -i\hbar / m \Delta t$. The local potential $U(x)$ plays obviously no role in this scaling law. The trajectories are almost always non-differentiable, a result already well known from the theory of Wiener-processes, the solution of the diffusion equation by path integration. One can check that any attempt to suppress this singularity and restore the differentiability even by an infinitesimal amount leads to the suppression of the right hand side of the canonical commutation relation $[x, p] = i\hbar$. The non-differentiability of the trajectory is not a mathematical artifact, it expresses somehow the very essence of Quantum Mechanics.

How about quantum field theory? Quantum Mechanics can be viewed as a quantum field theoretical model in 0+1 space-time dimensions. The integrand of the path integral for a free, massless scalar particle in dimensions d consists of the product of factors like

$$e^{-i \frac{a^{d-2}}{2} \Delta \phi^2} \quad (123)$$

The exponent is order one for

$$\Delta\phi \approx a^{1-\frac{d}{2}} = \begin{cases} \sqrt{\Delta t} & d = 0 + 1 \\ a^0 & d = 1 + 1 \\ \frac{1}{a} & d = 3 + 1 \end{cases} \quad (124)$$

The typical field configuration is nowhere continuous in quantum field theory, the discontinuity being finite in two dimensions and divergent beyond it. This surprising looking singular structure is actually the standard power counting in real space, as opposed to momentum space. It underlines the non-triviality and the fragility of any topological structure or argument in Quantum Field Theory.

Let us return to Quantum Mechanics to identify an interesting consequence of the UV singularities. The question we turn to is the meaning of the square of the velocity in the kinetic energy term in the path integral (121) when the trajectories are non-differentiable apart of a set of zero measure. Considering this question from the point of view of a quantum field theory in dimension 0+1 one starts to think in terms of the perturbation expansion, based on the free propagator

$$\langle T[x(t_1)x(t_2)] \rangle = \frac{i\hbar}{m} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1-t_2)}}{\omega^2 + i\epsilon}. \quad (125)$$

The average of the velocity square is then indeed linearly diverging

$$\frac{1}{T} \langle \int dt \left(\frac{dx(t)}{dt} \right)^2 \rangle = -\frac{1}{T} \langle \int dt x(t) \frac{d^2 x(t)}{dt^2} \rangle = -\frac{i\hbar}{m} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} \frac{d\omega}{2\pi} = -\frac{i\hbar}{m\Delta t} \quad (126)$$

as predicted by power counting. Let us now introduce an external vector potential, the Hamiltonian

$$H = \frac{(p - A(x))^2}{2m} + U(x) \quad (127)$$

leads to the path integral

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{N/2} \prod_{j=1}^{N-1} \int dx_j e^{\frac{i}{\hbar} \Delta t \sum_{n=1}^N \left[\frac{m}{2} \left(\frac{x_n - x_{n-1}}{\Delta t} \right)^2 - \frac{x_n - x_{n-1}}{\Delta t} A \left(\frac{1}{2}(x_n + x_{n-1}) + \eta(x_n - x_{n-1}) \right) - U(x_n) \right]}. \quad (128)$$

The new velocity coupling, represented by the vector potential may produce some surprise because the quantum fluctuations of the velocity are divergent. This is the reason that the η -parameter, a free parameter of the regularization is introduced. One expects no η -dependence according to the naive, classical picture where the trajectories are differentiable. But it is there,

$$\frac{d}{d\eta} \langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \frac{i}{\hbar} \Delta t \langle \int dt \frac{dx_j}{dt} \frac{dx_k}{dt} \partial_k A_j \rangle = \nabla \mathbf{A} \frac{i}{\hbar} \Delta t \left(-\frac{i\hbar}{m} \right) T \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} = \nabla \mathbf{A} \frac{T}{m} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{\Lambda} = \eta \frac{T}{m} \nabla \mathbf{A} \quad (129)$$

and we need $\eta = 0$, the mid-point prescription, in order to satisfy gauge invariance.

The reason of this peculiarity can be seen the clearest in the follow example. Let us consider the integral

$$I = \int_{t_i}^{t_f} dt \frac{f(x(t))}{dt}, \quad (130)$$

$f(x)$ being an analytic function. We obviously have

$$I = \sum_n [f(x_{n+1}) - f(x_n)] = f(x(t_f)) - f(x(t_i)). \quad (131)$$

We can rewrite the first equation by expanding around the base point $x_n^\eta = \frac{1}{2}(x_n + x_{n-1}) + \eta(x_n - x_{n-1})$, $x_n = x(t_n)$ in each interval

$$\begin{aligned} I &= \sum_n \left[f(x_n^\eta) + \Delta x_n \left(\frac{1}{2} - \eta \right) f'(x_n^\eta) + \frac{\Delta x_n^2}{2} \left(\frac{1}{2} - \eta \right)^2 f''(x_n^\eta) \right. \\ &\quad \left. - \left(f(x_n^\eta) - \Delta x_n \left(\frac{1}{2} + \eta \right) f'(x_n^\eta) + \frac{\Delta x_n^2}{2} \left(\frac{1}{2} + \eta \right)^2 f''(x_n^\eta) \right) + \mathcal{O}(\Delta x_n^3) \right] \\ &= \sum_n \Delta x_n f'(x_n^\eta) - \eta \sum_n \Delta x_n^2 f''(x_n^\eta) + \mathcal{O}(\Delta x^3) \end{aligned} \quad (132)$$

where $\Delta x_n = x_n - x_{n-1}$. What happens with the expectation value of the integral? It shows 'anomalous' behavior,

$$\begin{aligned} \left\langle \int_{t_i}^{t_f} dt \frac{f(x(t))}{dt} \right\rangle &= \left\langle \int_{t_i}^{t_f} dt \frac{dx(t)}{dt} \frac{df(x(t))}{dx} \right\rangle - \eta \left\langle \int_{t_i}^{t_f} dt \left(\frac{dx(t)}{dt} \right)^2 \frac{d^2 f(x(t))}{dx^2} \right\rangle \\ &= \left\langle \int_{t_i}^{t_f} dt \frac{dx(t)}{dt} \frac{df(x(t))}{dx} \right\rangle + \eta \frac{i\hbar}{m} \left\langle \int_{t_i}^{t_f} dt \frac{d^2 f(x(t))}{dx^2} \right\rangle. \end{aligned} \quad (133)$$

Integrals of this type are called Ito-integrals. The Ito-calculus contains unusual rules because one has to go one order more in the expansion in any variable transformation compared to the usual Riemann-integrals due to the fractal nature of the trajectories, dominating the path integral. Such a sensitivity on the regulator which is unexpected according to our classical intuition is called quantum anomaly.

The UV singularities are actually rather strong in Quantum Mechanics. The reason is that the degree of divergences increases with the number of external legs according to the power counting rule (18) below the lower critical dimension, $d < 2$. The BPHZ renormalization strategy is insufficient to remove the divergences. Quantum Mechanics is therefore non-renormalizable, but this poses no physical problem because it is supposed to be only an effective theory bounded by relativistic effects. We are not accustomed to see UV divergences in Quantum Mechanics because they are absent in the case of the Hamiltonian like (122) which mixes no non-commuting operators. Once the mixing between x and p is beyond quadratic order as in the case of the Hamiltonian (127) then the standard power counting predicts UV divergences which cannot be removed by the BPHZ strategy. The UV divergences are $\mathcal{O}(\hbar)$ and they modify certain relations of classical calculus, based on sufficiently differentiable curves and generate anomalies. In the standard quantum field theoretical interpretation the UV divergences for $d > 1$ are proportional not only with \hbar but with the number of degrees of freedom.

We see in this manner that the non-classical corrections to certain classical rules are not restricted to the breakdown of any symmetries. What may explain that anomaly was found in relation of symmetry is the the classical rules are clearer in the cases of a symmetry because the latter provides us non-trivial relations valid for any symmetrical dynamics and which can be tested against anomaly.

There are several 'accidental' features of the quantum anomaly:

1. It violates the rules of (classical) calculus.
2. It renders the naive continuum limit is wrong or non-trivial, eg. Eq. (129) in conventional three-dimensional Quantum Mechanics. The effect is there due to the presence of a cut-off, therefore anomalies provide observable evidences for a cut-off in realistic theories.
3. The best way to pin down the origin of the peculiarities is to realize that the integrand of the loop-integral is not uniformly convergent, $\lim_{\Lambda \rightarrow \infty} \int \frac{d\omega}{\Lambda} = 1 \neq \int \lim_{\Lambda \rightarrow \infty} \frac{d\omega}{\Lambda} = 0$, in the case of Eq. (129). It was noted at the beginning of section IC that the uniform convergence of the loop-integrals, the possibility of the interchanging the order of the removal of the cut-off and the loop-integration which is the base of the recursive removal of the sub-divergences. This may fail for anomalous loop-integral. But there is anomalies do not present a loophole to the BPHZ procedure due to the next peculiarity.
4. The power counting over estimates the degree of divergence for anomalous loop-integrals. Loop-integrals may have 'accidental' finiteness due to some cancellation. The loop-integral of Eq. (129) is supposed to be logarithmically divergent according to the power counting. But the logarithm is missing due to the simplicity of the integrand where some of the dimension is carried by the cut-off itself instead of a non-linear integrand.
5. Accidental finiteness of some loop-integrals in Quantum Field Theory may hide some free, adjustable coupling constants which play no role in canceling the divergences. Furthermore, the accidental finiteness identifies the vertices which are relevant but have no role to play in the UV divergent structure as explained at the beginning of this section.
6. Anomalies occur at infinitely short distances. Any cut-off theory where the regulator is explicitly kept is anomaly free in a trivial manner because it follows the standard rule of calculus. The evolution of the bare or the effective action is followed by the renormalization group equation. Once the cut-off is finite the evolution equation is anomaly-free. How can anomaly appear in this procedure? The ingredients of the dynamics between the gliding cut-off and infinity must be present in the initial conditions. Once they are put in they do not evolve anymore. This is the RG analogy of the non-renormalization theorem of Bardeen.
7. Anomalies are cut-off effects. Thought the regulator is represented by irrelevant operators as well the anomaly is independent of the actual choice of the regulator and obeys universality. This is a feature pointing to a global view of the renormalization group to be discussed later.

APPENDIX A: QUANTUM FIELD THEORY

Quantum Field Theory is about the physics of many equivalent particles. The simplest system, spinless particles without interaction has the total energy

$$E_{\text{tot}} = \sum_{\alpha} E(\mathbf{p}_{\alpha}) \quad (\text{A1})$$

where the index α labels the particles. It is better to reorganize this sum by collecting the contributions of particle belonging to the same momentum \mathbf{p} together,

$$E_{\text{tot}} = \sum_{\mathbf{p}} n(\mathbf{p})E(\mathbf{p}) \quad (\text{A2})$$

with $n(\mathbf{p})$ being the occupation number. Notice that each momentum sector has equidistant spectrum. The only one-dimensional quantum system with his property is the harmonic oscillator. Therefore, one introduces a harmonic oscillator for each momentum and the Fock-space where the physics of many-particle system is played out is the direct product of the corresponding Hilbert spaces, $\mathcal{H} = \otimes \prod_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}$. There is a destruction operator $a(\mathbf{p})$ corresponding to each momentum sector and this family can be viewed as a operator valued field variable given in the momentum space. Its real space analogy is the formal linear superposition of a Fourier integral,

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} a(\mathbf{p}), \quad (\text{A3})$$

called quantum field. This is taken as the generalized 'coordinate operator' of the system. Its canonical pair, $\pi(\mathbf{x})$, is supposed to satisfy the Heisenberg commutation relation

$$[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}'), \quad [\phi(\mathbf{x}), \phi(\mathbf{x}')] = [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0. \quad (\text{A4})$$

This single component real scalar field theory can be imagined as a system where the coordinates are labeled by the space points, $\phi(\mathbf{x})$. To render the mathematical expressions well defined one introduces a lattice in the coordinate space, $\mathbf{x} \rightarrow a\mathbf{n}$, where a is the lattice spacing and $\mathbf{n} = (n_1, n_2, n_3)$ is a vector with integer components, $0 \leq n_j \leq N$. The minimal distance of this theory, a , is called ultraviolet (UV) cutoff. The maximal distance, $L = Na$, the size of the quantization box is usually called the infrared (IR) cutoff. The continuum limit $a \rightarrow 0$, performed by keeping the physical content of the theory fixed. The removal of the IR cutoff, $L \rightarrow \infty$ is the thermodynamical limit. We take a small but finite a and a large but finite L at the time being. The coordinate of our system is the field configuration $\phi_{\mathbf{n}} = \phi(a\mathbf{n})$ which can be considered as a point in the N^3 -dimensional coordinate space.

The action is written as

$$S = \int d^4x L(\phi, \partial\phi) \quad (\text{A5})$$

in terms of the Lagrange which is assumed to be

$$L(\phi, \partial_{\mu}\phi) = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{m^2}{2}\phi^2 - U(\phi). \quad (\text{A6})$$

In lattice regularization the space-derivatives in the Lagrangian are given as

$$\nabla_j\phi(\mathbf{x}) = \frac{\phi_{\mathbf{n}+\mathbf{e}_j} - \phi_{\mathbf{n}}}{a}, \quad (\text{A7})$$

\mathbf{e}_j denoting the unit vector in the direction of j . The canonical momentum is defined in terms of the velocity by

$$\pi_{\mathbf{n}} = \frac{\partial L(\phi(x), \partial\phi(x))}{\partial\partial_0\phi(a\mathbf{n})}. \quad (\text{A8})$$

The quantization of the lattice regulated theory consists of replacing the canonical coordinates and momenta by operators and imposing the Heisenberg commutation relation

$$[\phi_{\mathbf{n}}, \pi_{\mathbf{n}'}] = i\delta_{\mathbf{n},\mathbf{n}'}, \quad [\phi_{\mathbf{n}}, \phi_{\mathbf{n}'}] = [\pi_{\mathbf{n}}, \pi_{\mathbf{n}'}] = 0 \quad (\text{A9})$$

which can be satisfied by

$$\pi_{\mathbf{n}} = \frac{1}{i} \frac{\partial}{\partial \phi_{\mathbf{n}}}. \quad (\text{A10})$$

The Hamiltonian, defined by

$$H = a^3 \sum_{\mathbf{n}} \partial_0 \phi_{\mathbf{n}} \pi_{\mathbf{n}} - L_t[\phi_{\mathbf{n}}, \partial_0 \phi_{\mathbf{n}}, \nabla \phi_{\mathbf{n}}]. \quad (\text{A11})$$

turns out to be

$$H = a^3 \sum_{\mathbf{n}} \left[\frac{1}{2} \pi_{\mathbf{n}} \pi_{\mathbf{n}} + \frac{1}{2a^2} \sum_{j=1}^3 (\phi_{\mathbf{n}} - \phi_{\mathbf{n}-j})^2 + \frac{m^2}{2} \phi_{\mathbf{n}}^2 + U(\phi_{\mathbf{n}}) \right]. \quad (\text{A12})$$

The construction of the momentum for continuous space starts with the definition of the functional derivative $\frac{\delta}{\delta \phi_{\mathbf{n}}}$, replacing the partial derivative in the definition of the canonical momentum,

$$\frac{\delta}{\delta \phi(a\mathbf{n})} = \lim_{a \rightarrow 0} \frac{1}{a^d} \frac{\partial}{\partial \phi_{\mathbf{n}}}. \quad (\text{A13})$$

The factor $1/a^d$ is needed in this expression to reproduce the relation

$$\frac{\delta}{\delta \phi(x)} \int d^d y f(y) \phi(y) = \lim_{a \rightarrow 0} \frac{1}{a^d} \frac{\partial}{\partial \phi_{\mathbf{n}}} a^d \sum_{\mathbf{n}'} f_{\mathbf{n}'} \phi_{\mathbf{n}'} = \phi_{\mathbf{n}} = \phi(x) \quad (\text{A14})$$

which is reasonable to impose on a functional derivative. The important lesson if this factor is that the definition of the functional derivative depends on the number of variables of the functions it is acting upon. The canonical momentum,

$$\pi(\mathbf{x}) = \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})}, \quad (\text{A15})$$

defined in the Lagrangian formalism as

$$\pi(\mathbf{x}) = \frac{\delta}{\delta \partial_0 \phi(x)} \int d^3 x L(\phi, \partial \phi) = \frac{\partial L(\phi(x), \partial \phi(x))}{\partial \partial_0 \phi(x)} \quad (\text{A16})$$

satisfies the commutation relation (A4).

This formalism leads to too difficult problems in functional analysis. But the difficulties lie in the evaluation of matrix elements rather than writing down formal expressions for the desired observables. The path integral representation of matrix elements, introduced below, allows us to write down the desired quantities without referring to operators. The price is to obtain the result in terms of integrals over a large number of variables. Naturally, the equivalence of the two formalisms assures that each mathematical problem or issue at one case corresponds to something similar at the other case. But we possess more analytical and numerical methods to deal with integrals with large number of variables than for problems in functional analysis.

The path integral method will be introduced first in the context of Quantum Mechanics and will after be generalized for Quantum Field Theory.

1. Path integral in nonrelativistic Quantum Mechanics

We shall work with a one dimensional nonrelativistic particle described by the Hamiltonian

$$H = \frac{p^2}{2m} + U(x) \quad (\text{A17})$$

with $[x, p] = i\hbar$.

Transition amplitudes: The propagator or transition amplitude between coordinate eigenstates $|x\rangle$ is

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle. \quad (\text{A18})$$

As a warmup exercise we first construct the path integral expression for (A18). The properties and the use of the path integral for matrix elements and expectation values will be discussed later.

The amplitude (A18) is a complicated function of the variables t , x_i and x_f . We simplify the problem by computing it for short time when it takes a simpler form and by constructing the finite time transition amplitude from the short time one. This latter step is accomplished by writing

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \langle x_f | \left(e^{-\frac{i}{\hbar} H \Delta t} \right)^N | x_i \rangle \quad (\text{A19})$$

with $\Delta t = t/N$ and inserting a resolution of the identity

$$\mathbb{1} = \int dx |x\rangle \langle x| \quad (\text{A20})$$

between each operator,

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \prod_{j=1}^{N-1} \int dx_j \langle x_j | e^{-\frac{i}{\hbar} H \Delta t} | x_{j-1} \rangle \quad (\text{A21})$$

where $x_0 = x_i$ and $x_N = x_f$. This relation which holds for any N becomes a path integral as $N \rightarrow \infty$. In fact, any trajectory between the given initial and final point can be approximated by a piecewise constant function when the length of the time interval Δt when the function is constant tends to zero.

In order to turn the simple path integral expression (A21) into something useful we need a simple approximation for the short time transition amplitudes. There are $\mathcal{O}(N)$ of them multiplied together therefore it is enough to have $\mathcal{O}(N^{-1}) = \mathcal{O}(\Delta t)$ accuracy in obtaining them. The first guess would be

$$\begin{aligned} \langle x | e^{-\frac{i}{\hbar} H \Delta t} | x' \rangle &\approx \langle x | 1 - \frac{i}{\hbar} H \Delta t | x' \rangle \\ &= \langle x | x' \rangle \left(1 - \frac{i \Delta t}{\hbar} \frac{\langle x | H | x' \rangle}{\langle x | x' \rangle} \right) \\ &\approx e^{\frac{i}{\hbar} \Delta t \frac{\langle x | H | x' \rangle}{\langle x | x' \rangle}} \end{aligned} \quad (\text{A22})$$

but the problem is the orthogonality of the basis vectors $\langle x | x' \rangle = \delta(x - x')$. In fact, the small parameter in the expansion is $\Delta t / \langle x | x' \rangle$ which is diverging for $x \neq x'$. To avoid this problem we use two overlapping basis in an alternating manner. In case of continuous space the choice of the other, overlapping basis is rather natural. It will be a momentum basis, $|p\rangle$ with $p|q\rangle = q|q\rangle$. The corresponding resolution of the identity,

$$1 = \int \frac{dp}{2\pi} |p\rangle \langle p|, \quad (\text{A23})$$

inserted in Eqs. (A22) yields

$$\begin{aligned} \langle x | e^{-\frac{i}{\hbar} H \Delta t} | x' \rangle &= \int \frac{dp}{2\pi} \langle x | e^{-\frac{i}{\hbar} H \Delta t} | p \rangle \langle p | x' \rangle \\ &\approx \int \frac{dp}{2\pi} \langle x | 1 - \frac{i}{\hbar} H \Delta t | p \rangle \langle p | x' \rangle \\ &= \int \frac{dp}{2\pi} \langle x | p \rangle \langle p | x' \rangle \left(1 - \frac{i \Delta t}{\hbar} \frac{\langle x | H | p \rangle}{\langle x | p \rangle} \right) \\ &\approx \int \frac{dp}{2\pi} e^{\frac{i}{\hbar} p(x-x') - \frac{i}{\hbar} \Delta t H(p,x)} \end{aligned} \quad (\text{A24})$$

with

$$H(p, x) = \frac{\langle x | H | p \rangle}{\langle x | p \rangle} = \frac{p^2}{2m} + U(x). \quad (\text{A25})$$

By replacing this expression into Eq. (A21) we arrive at a path integral in configuration space,

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int dx_j \prod_{k=1}^N \int \frac{dp_k}{2\pi} e^{\frac{i}{\hbar} \Delta t \sum_{\ell=1}^N [p_\ell \frac{x_\ell - x_{\ell-1}}{\Delta t} - H(p_\ell, x_\ell)]} \quad (\text{A26})$$

which can be written in a condensed, formal notation as

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \int_{x(0)=x_i}^{x(t)=x_f} D[x] \int D[p] e^{\frac{i}{\hbar} \int d\tau [p(\tau) \dot{x}(\tau) - H(p(\tau), x(\tau))]} \quad (\text{A27})$$

by suppressing the cutoff Δt . Notice that the integration over coordinate or momentum trajectories of fixed or free initial and final points, respectively.

The integrand in the exponent would be the Lagrangian in classical mechanics. We can not rely on this analogy in Quantum Mechanics but can carry out the momentum integral. The use of the Fresnel integral

$$\int_{-\infty}^{\infty} dx e^{i \frac{a}{2} x^2 + i b x} = \sqrt{\frac{2\pi}{|a|}} e^{-i \frac{b^2}{2a}} \times \begin{cases} \sqrt{i} & a > 0 \\ \frac{1}{\sqrt{i}} & a < 0 \end{cases} \quad (\text{A28})$$

gives the path integral

$$\begin{aligned} \langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int dx_j \\ &\times e^{\frac{i}{\hbar} \Delta t \sum_{\ell=1}^N [\frac{m}{2} (\frac{x_{\ell} - x_{\ell-1}}{\Delta t})^2 - U(x_{\ell})]} \end{aligned} \quad (\text{A29})$$

in coordinate space which reads in condensed notation

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \int_{x(0)=x_i}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int d\tau L(x(\tau), \dot{x}(\tau))} \quad (\text{A30})$$

with

$$L = \frac{m}{2} \dot{x}^2 - U(x). \quad (\text{A31})$$

The expressions (A27) and (A30) are easy to memorize but are formal because the functional integration measure is defined by a limiting procedure, spelled out in the more involved expressions (A26) and (A29). The integration over the momentum recovers the Lagrangian (A31) from the Hamiltonian (A17). Such an agreement with the Legendre transformation of Classical Mechanics is restricted to the Gaussian integration, ie. Hamiltonians of the form (A17), ie. which are quadratic in the momentum and contain the dependence in the coordinate in an additive term.

We consider now two important but trivial generalizations of the formulae obtained so far. First we replace the coordinate eigenstates by arbitrary initial and final states,

$$\begin{aligned} \langle \psi_f | e^{-\frac{i}{\hbar} H t} | \psi_i \rangle &= \int dx_i dx_f \psi_f^*(x_f) \langle \psi_f | e^{-\frac{i}{\hbar} H t} | \psi_i \rangle \psi_i(x_i) \\ &= \int D'[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} \end{aligned} \quad (\text{A32})$$

where

$$\int D'[x] = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int dx_i dx_f \psi_f^*(x_f) \psi_i(x_i) \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int dx_j. \quad (\text{A33})$$

Second, we generalize the results for particle moving in d -dimensional space. The steps leading to Eq. (A26) can easily be repeated in this case leading to

$$\langle \mathbf{x}_f | e^{-\frac{i}{\hbar} H t} | \mathbf{x}_i \rangle = \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int d^d x_j \prod_{k=1}^N \int \frac{d^d p_k}{(2\pi)^d} e^{\frac{i}{\hbar} \Delta t \sum_{\ell=1}^N [\mathbf{p}_{\ell} \cdot \frac{\mathbf{x}_{\ell} - \mathbf{x}_{\ell-1}}{\Delta t} - H(\mathbf{p}_{\ell}, \mathbf{x}_{\ell})]} \quad (\text{A34})$$

with

$$H = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}) \quad (\text{A35})$$

which takes the form

$$\langle \mathbf{x}_f | e^{-\frac{i}{\hbar} H t} | \mathbf{x}_i \rangle = \int_{\mathbf{x}(0)=\mathbf{x}_i}^{\mathbf{x}(t)=\mathbf{x}_f} D[\mathbf{x}] \int D[\mathbf{p}] e^{\frac{i}{\hbar} \int d\tau [\mathbf{p}(\tau) \dot{\mathbf{x}}(\tau) - H(\mathbf{p}(\tau), \mathbf{x}(\tau))]} \quad (\text{A36})$$

in condensed notation. The Lagrangian path integral reads in d -dimensions as

$$\begin{aligned} \langle \mathbf{x}_f | e^{-\frac{i}{\hbar} H t} | \mathbf{x}_i \rangle &= \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{d}{2}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{d}{2}} \int d^d x_j \\ &\quad \times e^{\frac{i}{\hbar} \Delta t \sum_{\ell=1}^N \left[\frac{m}{2} \left(\frac{\mathbf{x}_\ell - \mathbf{x}_{\ell-1}}{\Delta t} \right)^2 - U(\mathbf{x}_\ell) \right]} \\ &= \int_{\mathbf{x}(0)=\mathbf{x}_i}^{\mathbf{x}(t)=\mathbf{x}_f} D[x] e^{\frac{i}{\hbar} \int d\tau L(\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau))} \end{aligned} \quad (\text{A37})$$

with

$$L = \frac{m}{2} \dot{\mathbf{x}}^2 - U(\mathbf{x}). \quad (\text{A38})$$

Matrix elements: The path integral representation of the transition amplitude generates the matrix elements of observables, too. We shall use the Heisenberg representation in this and the following section and start with the observable $F(x(t'))$, $0 < t' < t$ where $F(x)$ is an arbitrary function. The matrix element

$$\begin{aligned} &\langle x_f | T[F(x(t')) e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle \\ &= \langle x_f | e^{-\frac{i}{\hbar} \int_{t'}^t d\tau H(\tau)} F(x(t')) e^{-\frac{i}{\hbar} \int_0^{t'} d\tau H(\tau)} | x_i \rangle \\ &= \int dx'_1 dx'_2 \langle x_f | e^{-\frac{i}{\hbar} \int_{t'}^t d\tau H(\tau)} | x'_1 \rangle \underbrace{\langle x'_1 | F(x) | x'_2 \rangle}_{\delta(x'_1 - x'_2) F(x'_1)} \langle x'_2 | e^{-\frac{i}{\hbar} \int_0^{t'} d\tau H(\tau)} | x_i \rangle \\ &= \int dx' \langle x_f | e^{-\frac{i}{\hbar} \int_{t'}^t d\tau H(\tau)} | x' \rangle F(x') \langle x' | e^{-\frac{i}{\hbar} \int_0^{t'} d\tau H(\tau)} | x_i \rangle \\ &= \int dx' \int_{x(t')=x'}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} F(x') \int_{x(0)=x_i}^{x(t')=x'} D[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} \\ &= \int_{x(0)=x_i}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} F(x(t')) \end{aligned} \quad (\text{A39})$$

where it was used in the last equation that the integration over trajectories from $x(0) = x_i$ to $x(t') = x'$ and from $x(t') = x'$ to $x(t) = x_f$ together with the integration over x' is equivalent with the integration over trajectories from $x(0) = x_i$ to $x(t) = x_f$. One has in a similar manner

$$\begin{aligned} &\langle x_f | T[F_1(x(t_1)) F_2(x(t_2)) e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle \\ &= \int_{x(0)=x_i}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} F_1(x(t_1)) F_2(x(t_2)) \end{aligned} \quad (\text{A40})$$

and in general, for an arbitrary functional $F[x]$ we find

$$\langle x_f | T[F[x] e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle = \int_{x(0)=x_i}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int dt L(x(t), \dot{x}(t))} F[x]. \quad (\text{A41})$$

The simple c-number multiplication is clearly the only reasonable definition of products in the path integral where no operators appear. The relations obtained above indicate that the "natural" multiplication law for time dependent problems in Quantum Mechanics is the time ordering because it is this product which goes over the c-number multiplication in the path integral representation.

One can formally rewrite Eq. (A41) for entire analytic functionals as

$$\langle x_f | T[F[x] e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle = F \left[\frac{\hbar}{i} \frac{\delta}{\delta j} \right] Z[j] |_{j=0} \quad (\text{A42})$$

by means of the generating functional

$$Z[j] = \int_{x(0)=x_i}^{x(t)=x_f} D[x] e^{\frac{i}{\hbar} \int dt [L(x(t), \dot{x}(t)) + x(t)j(t)]}. \quad (\text{A43})$$

In fact, to prove Eq. (A42) we use the functional Taylor expansion

$$\begin{aligned} & \langle x_f | T[F[x] e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^n \int dt_j \frac{\delta^n F[x]}{\delta x(t_1) \cdots \delta x(t_n)} \langle x_f | T[\prod_{j=1}^n x(t_j) e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^n \int dt_j \frac{\delta^n F[x]}{\delta x(t_1) \cdots \delta x(t_n)} \prod_{j=1}^n \frac{\hbar}{i} \frac{\delta}{\delta j(t_j)} Z[j]_{|j=0}. \end{aligned} \quad (\text{A44})$$

Matrix elements of observables composed by the momentum can be calculated in a similar manner. We give the general result for a mixed observable $F[p, x]$ only,

$$\langle x_f | T[F[p, x] e^{-\frac{i}{\hbar} \int_0^t d\tau H(\tau)}] | x_i \rangle = F \left[\frac{\hbar}{i} \frac{\delta}{\delta j}, \frac{\hbar}{i} \frac{\delta}{\delta k} \right] Z[j, k]_{|j=k=0} \quad (\text{A45})$$

where the generating functional

$$Z[j, k] = \int_{x(0)=x_i}^{x(t)=x_f} D[x] D[p] e^{\frac{i}{\hbar} \int dt [p(t)\dot{x}(t) - H(p(t), x(t)) + x(t)j(t) + p(t)k(t)]} \quad (\text{A46})$$

induces all coordinate and momentum insertions in the matrix elements. It is easy to integrate over the momentum for the Hamiltonian (A17),

$$\begin{aligned} Z[j, k] &= \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int dx_j \prod_{k=1}^N \int \frac{dp_k}{2\pi} e^{\frac{i\Delta t}{\hbar} \sum_{\ell=1}^N [p_{\ell} \frac{x_{\ell} - x_{\ell-1}}{\Delta t} - \frac{p_{\ell}^2}{2m} - U(x_{\ell}) + x_{\ell} j_{\ell} + p_{\ell} k_{\ell}]} \\ &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int dx_j \\ &\quad \times e^{\sum_{\ell=1}^N [\frac{im}{2\Delta t \hbar} (x_{\ell} - x_{\ell-1} + \Delta t k_{\ell})^2 - \frac{i\Delta t}{\hbar} U(x_{\ell}) + x_{\ell} j_{\ell}]} \\ &= \int D[x] e^{\frac{i}{\hbar} \int d\tau [\frac{m}{2} \dot{x}^2(\tau) + \frac{m}{2} k^2(\tau) + m\dot{x}(\tau)k(\tau) - U(x(\tau)) + x(\tau)j(\tau)]}. \end{aligned} \quad (\text{A47})$$

Perturbation expansion: The path integral is exactly calculable for quadratic actions, describing non-interacting particles. The simplest way to deal with interaction is the perturbation expansion. It is actually much more simple than in the operator formalism because we face only c-numbers in the path integrals. By assuming $U(x) \approx 0$ we have

$$\begin{aligned} \langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle &= \int D[x] e^{\frac{i}{\hbar} \int dt' \frac{m}{2} \dot{x}^2(t') - U(x(t'))} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n! \hbar^n} \prod_{j=1}^n \int dt_j \int D[x] e^{\frac{i}{\hbar} \int dt' \frac{m}{2} \dot{x}^2(t')} \prod_{j=1}^n U(x(t_j)) \end{aligned} \quad (\text{A48})$$

which can be written according to Eq. (A41) as

$$\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n! \hbar^n} \prod_{j=1}^n \int dt_j \langle x_f | T[\prod_{j=1}^n U(x(t_j)) e^{-\frac{i}{\hbar} \int dt' H(t')}] | x_i \rangle. \quad (\text{A49})$$

These equations show that the propagation in the presence of a potential can be viewed as a sequence of interactions with the potential separated by free propagation.

2. Path integral in Quantum Field Theory

We shall look for the matrix elements of the time evolution operator

$$\langle \phi_f | e^{-iHt} | \phi_i \rangle \quad (\text{A50})$$

($c = \hbar = 1$) for the field eigenstates

$$\phi_{\mathbf{n}} | \phi \rangle = \phi_{\mathbf{n}} | \phi \rangle. \quad (\text{A51})$$

The resolutions of the identity

$$\mathbb{1} = \prod_{\mathbf{n}} \int d\phi_{\mathbf{n}} | \phi \rangle \langle \phi | = \prod_{\mathbf{n}} \int d\pi_{\mathbf{n}} | \pi \rangle \langle \pi | \quad (\text{A52})$$

with

$$\pi_{\mathbf{n}} | \pi \rangle = \pi_{\mathbf{n}} | \pi \rangle \quad (\text{A53})$$

and

$$\langle \phi | \pi \rangle = \frac{1}{(2\pi)^{N^3}} e^{ia^3 \sum_{\mathbf{n}} \phi_{\mathbf{n}} \pi_{\mathbf{n}}} \quad (\text{A54})$$

give

$$\langle \phi_f | e^{-iH\Delta t} | \phi_i \rangle \approx \prod_{\mathbf{n}} \int \frac{d\pi_{\mathbf{n}}}{2\pi} e^{i \sum_{\mathbf{n}} \pi_{\mathbf{n}} (\phi_{\ell, \mathbf{n}} - \phi_{\ell-1, \mathbf{n}}) - i\Delta t H(\pi_{\ell}, \phi_{\ell})} \quad (\text{A55})$$

with

$$H(\pi, \phi) = \frac{\langle \phi | H | \pi \rangle}{\langle \phi | \pi \rangle} = a^3 \sum_{\mathbf{n}} \left[\frac{1}{2} \pi_{\mathbf{n}} \pi_{\mathbf{n}} + \frac{1}{2a^2} \sum_{j=1}^3 (\phi_{\mathbf{n}+j} - \phi_{\mathbf{n}})^2 + \frac{m^2}{2} \phi^2(\mathbf{x}) + U(\phi(\mathbf{x})) \right]. \quad (\text{A56})$$

The integration over the momentum yields

$$\begin{aligned} \langle \phi_f | e^{-iHt} | \phi_i \rangle &= (2\pi i \Delta t)^{-\frac{N^3}{2}} \lim_{N_t \rightarrow \infty} \prod_{j=1}^{N_t-1} (2\pi i \Delta t)^{-\frac{N^3}{2}} \prod_{\mathbf{n}} \int d\phi_{\mathbf{n}} \\ &\quad \times e^{ia^3 \Delta t \sum_{\ell=1}^{N_t} \sum_{\mathbf{n}} \left[\frac{1}{2\Delta t^2} (\phi_{\ell, \mathbf{n}} - \phi_{\ell-1, \mathbf{n}})^2 + \frac{1}{2a^2} \sum_{j=1}^3 (\phi_{\ell, \mathbf{n}+j} - \phi_{\ell, \mathbf{n}})^2 - \frac{m^2}{2} \phi_{\ell, \mathbf{n}}^2 - U(\phi_{\ell, \mathbf{n}}) \right]} \\ &= \int D[\phi] e^{iS[\phi]} \end{aligned} \quad (\text{A57})$$

involving the action

$$S[\phi] = \Delta t a^3 \sum_{\ell=1}^{N_t} \sum_{\mathbf{n}} \left[\frac{1}{2\Delta t^2} (\phi_{\ell, \mathbf{n}} - \phi_{\ell-1, \mathbf{n}})^2 + \frac{1}{2a^2} \sum_{j=1}^3 (\phi_{\ell, \mathbf{n}+j} - \phi_{\ell, \mathbf{n}})^2 - \frac{m^2}{2} \phi_{\ell, \mathbf{n}}^2 - U(\phi_{\ell, \mathbf{n}}) \right] \quad (\text{A58})$$

which will be denoted in the continuum limit by

$$S[\phi] \rightarrow \int d^4x \left[\frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) - \frac{m^2}{2} \phi^2(x) - U(\phi(x)) \right]. \quad (\text{A59})$$

But the observables must be calculated with finite regulator and the continuum limit, $a \rightarrow 0$, is carried out at the end only. The trace of the time evolution operator is obtained by integrating over the same initial and final field configuration and can be written as

$$\text{Tr} e^{-iHt} = \int D[\phi] e^{iS[\phi]} \quad (\text{A60})$$

where the path integral is over periodic field configurations in time.

3. Observables and Green functions

It is well known that we can not measure the coordinates and the momenta simultaneously in Quantum Mechanics. But there is no problem in measuring the coordinate or the momenta alone. When we are interested in high energy phenomena then the precision of the measurement of either the coordinate or the momenta becomes limited. The localization can not be measured with accuracy much better than the Compton wave length of the particle due to the pair-creation process going on as the result of our attempt to localize the particle. The high energy processes take place in short time and the uncertainty of determining the frequency of an oscillation seen for a short time introduces the limitation in the measurement of energy and momenta. One is forced to focus the attention on scattering transition probabilities which remain well defined at arbitrary high energies.

The reduction formulae, a breakthrough in the history of Quantum Field Theory, assures that the transition amplitudes are proportional to the singular term of the connected Green functions due to the mass-shell singularities. The Green function of order n is defined by the expectation value

$$iG^{(n)}(x_1, \dots, x_n) = \langle 0|T[\phi_H(x_1) \cdots \phi_H(x_n)]|0\rangle_H \quad (\text{A61})$$

in the Heisenberg representation. Thus the usual goal in Quantum Field Theory is the calculation of the Green functions.

We are interested in these lectures in the UV finiteness and the scale dependence of the Green functions. Most of these question can be studied when the time is analytically continued to imaginary values to simplify the convergences of loop-integrals.

APPENDIX B: PERTURBATION EXPANSION

1. Green functions

The functional formalism is advantageous because it allows us to handle infinitely many Green function in the same time. The generating functional for the Green functions is defined as the vacuum-to-vacuum transition amplitude in the presence of an external source $j(x)$ coupled linearly to the field operator in the action,

$$S[\phi] \rightarrow S[\phi] + \int dx j(x)\phi(x) = S[\phi] + j \cdot \phi, \quad (\text{B1})$$

leading to the path integral expression

$$Z[j] = \langle 0|S|0\rangle_j = \int D[\phi] e^{iS[\phi] + ij \cdot \phi}, \quad (\text{B2})$$

where the functions on the space-time are viewed as vectors and the scalar product denotes integration over the coordinates. The name is justified by the functional Taylor-expansion

$$Z[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n iG(x_1, \dots, x_n) j(x_1) \cdots j(x_n) \quad (\text{B3})$$

containing the interacting Green functions. The convergence radius of this series is believed to be non-zero for space-time dimensions $d > 3$ because the non-analytic source dependence should come from the creation of bound states which starts at a finite threshold for the strength of the external source above two space dimensions. Thus the variable j of the generating functional is a formal device only which is kept infinitesimal and is introduced to facilitate the book-keeping.

The generator functional can easily be obtained for a free system with action

$$S_0[\phi] = -\frac{1}{2} \int dx \phi(x)(\square + m^2)\phi(x) = \frac{1}{2} \phi \cdot G_0^{-1} \cdot \phi. \quad (\text{B4})$$

In fact,

$$Z_0[j] = \int D[\phi] e^{\frac{i}{2} \phi \cdot G_0^{-1} \cdot \phi + ij \cdot \phi} = e^{-\frac{i}{2} j \cdot G_0 \cdot j}. \quad (\text{B5})$$

The weak form of the Wick theorem, expressing the free Green functions $G_0(x_1, \dots, x_n)$ in terms of the free propagators $G_0(x, y)$ can be obtained by starting with the identity

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n iG_0(x_1, \dots, x_n) j(x_1) \cdots j(x_n) = e^{-\frac{i}{2} j \cdot G_0 \cdot j}, \quad (\text{B6})$$

and identifying the $2k$ -th order functional derivatives of both sides at $j = 0$,

$$iG_0(x_1, \dots, x_{2k}) = \frac{i^k}{2^k k!} \sum_{\pi \in S_{2k}} G_0(x_{\pi(1)}, x_{\pi(2)}) \cdots G_0(x_{\pi(2k-1)}, x_{\pi(2k)}), \quad (\text{B7})$$

where the summation is over the symmetric group of order $2k$, ie. the permutations of $2k$ objects. The Green functions of order odd are vanishing.

The perturbation series for $Z[j]$ is obtained by separating off the interaction part in the action, $S[\phi] = S_0[\phi] + S_i[\phi]$ and expanding the integrand in (B2) in the interaction,

$$\begin{aligned} Z[j] &= \int D[\phi] e^{\frac{i}{2} \phi \cdot G_0^{-1} \cdot \phi + S_i[\phi] + i j \cdot \phi} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int D[\phi] e^{\frac{i}{2} \phi \cdot G_0^{-1} \cdot \phi + i j \cdot \phi} S_i^n[\phi]. \end{aligned} \quad (\text{B8})$$

The last term, representing the interaction can formally be factorized of the path integral by means of the generating functional for the free Green functions,

$$\begin{aligned} Z[j] &= \sum_{n=0}^{\infty} \frac{i^n}{n!} S_i^n \left[\frac{1}{i} \frac{\delta}{\delta \phi} \right] \int D[\phi] e^{\frac{i}{2} \phi \cdot G_0^{-1} \cdot \phi + i j \cdot \phi} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} S_i^n \left[\frac{1}{i} \frac{\delta}{\delta \phi} \right] Z_0[j]. \end{aligned} \quad (\text{B9})$$

The resummation of the series finally yields

$$Z[j] = e^{i S_i^n \left[\frac{1}{i} \frac{\delta}{\delta \phi} \right]} Z_0[j] = e^{i S_i^n \left[\frac{1}{i} \frac{\delta}{\delta \phi} \right]} e^{-\frac{i}{2} j \cdot G_0 \cdot j}. \quad (\text{B10})$$

Let us consider the theory defined by the Lagrangian (A6) with $U(\phi) = g/4! \phi^4$,

$$\begin{aligned} Z[j] &= e^{-\frac{ig}{4!} \int dx \frac{\delta^4}{\delta \phi^4(x)} e^{-\frac{i}{2} j \cdot G_0 \cdot j}} \\ &= \left[1 - \frac{ig}{4!} \int dx \frac{\delta^4}{\delta \phi^4(x)} + \frac{(ig)^2}{2!(4!)^2} \int dx_1 dx_2 \frac{\delta^4}{\delta \phi^4(x_1)} \frac{\delta^4}{\delta \phi^4(x_2)} + \cdots \right] e^{-\frac{i}{2} j \cdot G_0 \cdot j}. \end{aligned} \quad (\text{B11})$$

The resulting perturbation series for the interacting Green functions,

$$\begin{aligned} iG(x, y) &= -\frac{\delta^2 Z[j]}{\delta j(x) \delta j(y)} \Big|_{j=0} \\ &= iG_0(x, y) - \frac{ig}{4} iG_0(x, y) \int dz (iG_0(z, z))^2 - \frac{ig}{2} \int dz iG_0(x, z) iG_0(z, z) iG_0(z, y) + \mathcal{O}(g^2) \end{aligned} \quad (\text{B12})$$

etc. Each contribution in the last line can be represented by a Feynman graphs.

2. Connected Green functions

The series in the second line of Eqs. (B9) expresses the generating functional as a sum over graphs, connected and disconnected. We would like to separate off the connected graphs. For this end we consider the functional derivative of the generator functional (B3),

$$\frac{\delta Z[j]}{\delta i j(x)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n iG(x, x_1, \dots, x_n) j(x_1) \cdots j(x_n). \quad (\text{B13})$$

This is a sum of graphs, some of them connected, others have disconnected components. Let us denote the sum of the connected graphs with a single leg at the space-time point x by $i\delta W[j]/\delta j(x)$. These graphs are present on the right hand side of Eq. (B13) and can be factorized out. The remaining graphs cover all graphs contributing to $Z[j]$. The symmetry prefactor is the product of those of the disconnected components of the graph therefore these graphs has the same prefactor than those in $Z[j]$. Therefore these graphs sum up to $Z[j]$. The integration of the relation

$$\frac{\delta Z[j]}{\delta j(x)} = i \frac{\delta W[j]}{\delta j(x)} Z[j] \quad (\text{B14})$$

gives

$$Z[j] = e^{iW[j]} \quad (\text{B15})$$

after having taken into account the initial condition $W[0] = 0$. The functional $W[j]$ introduced in this manner generates the connected Green functions G_c , obtained by omitting the disconnected graphs in the perturbation series,

$$W[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n i G_c(x_1, \dots, x_n) j(x_1) \cdots j(x_n). \quad (\text{B16})$$

The importance of the connected Green functions is that the usual Green functions can be written as sum of products where each factor in the products is a connected Green function.

3. One-particle irreducible Green functions

The one-particle irreducible (1PI) graph play an important role in quantum field theory. A connected graph is called one-particle irreducible if no cutting of any internal line makes two disconnected graphs. We remove the external legs by multiplying the original graphs by the inverse of the propagators at each leg. This step amounts to the removal of the mass-shell singularity and the value of the resulting graph represents a contribution to a scattering amplitude. The 1PI graphs are called vertex functions because there 1PI structure allows us to identify them with elementary coupling strengths, controlling the scattering processes. This identification is particularly visible in the skeleton expansion where all radiative corrections attached at a given line are summed up. After this partial resummation of the perturbation series the Green function is a sum of products where a product contains vertices and propagators. The former is represented by the irreducible Green functions and the latter by the full propagator.

Another use of the 1PI graph is that the UV divergence of a one-particle reducible graph can be factorized into the product of diverging factors corresponding to the 1PI parts of the graphs.

The 1PI Green functions, $\Gamma(x_1, \dots, x_n)$, obtained from the connected Green functions by omitting the one particle reducible graphs is called the effective action

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma(x, x_1, \dots, x_n) \Phi(x_1) \cdots \Phi(x_n) \quad (\text{B17})$$

and is defined by

$$e^{i\Gamma[\Phi] + i\Phi \cdot j} = \int D[\phi] e^{iS[\phi] + i\phi \cdot j} \quad (\text{B18})$$

where

$$\Phi(x) = \langle 0 | \phi(x) | 0 \rangle_j = \frac{1}{Z[j]} \frac{\delta Z[j]}{\delta i j(x)} = \frac{\delta W[j]}{\delta j(x)} \quad (\text{B19})$$

is the field expectation value in the ground state in the presence of a given external source. Note that $\Phi(x)$ is a rather complicated function of the source $j(x)$ and it becomes the physical expectation value for $j = 0$ only. The physical interpretation of the effective action according to Eq. (B18) is that it gives the contribution of the quantum dynamics to the phase of the vacuum-to-vacuum transition amplitude as the function of the field expectation value after the separation of the source term. Eq. (B18) reads

$$\Gamma[\Phi] + \Phi \cdot j = W[j], \quad (\text{B20})$$

in a more compact form which together with the last equation in Eqs. (B19) show that the effective action is the functional Legendre transform of $W[j]$. The proof that it generates the 1PI Green functions is rather involved and is based on recursion.

We note here only that the functional derivative of Eq. (B20) with respect to Φ ,

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} + j(x) + \int dy \Phi(y) \frac{\delta j(y)}{\delta\Phi(x)} = \int dy \frac{\delta W[j]}{\delta j(y)} \frac{\delta j(y)}{\delta\Phi(x)} \quad (\text{B21})$$

shows that the inverse Legendre transform is given by Eq. (B20) and

$$j(x) = -\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}. \quad (\text{B22})$$

4. One-loop effective potential of the ϕ^4 model.

We shall demonstrate the use of the perturbation expansion for the calculation of the leading order contribution to the effective potential, the local part of the effective action, in the ϕ^4 model in $d = 4$ Euclidean dimensions, defined by the Lagrangian

$$L = \frac{1}{2}(\partial\phi)^2 + U_B(\phi), \quad (\text{B23})$$

with

$$U_B(\phi) = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4. \quad (\text{B24})$$

Let us start with the path integral

$$e^{\frac{1}{\hbar}W} = \int D[\phi] e^{-\frac{1}{\hbar} \int_x [\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4]} \quad (\text{B25})$$

where Planck constant is reintroduced. It multiplies the free propagators with \hbar and the vertices with $1/\hbar$ in the graphs. Therefore a graph of $\mathcal{O}(g^n)$ is $\mathcal{O}(\hbar^{I-n})$ where I denotes the number of internal lines. The relation $L = I - n + 1$ gives the number of independent momentum variables of the internal lines of the graph as the difference of the total number of internal lines and the number of energy-momentum conservation laws. Therefore, the graphs contributing to W are $\mathcal{O}(g^L)$, the expansion in the Planck constant organizes the graphs according to their number of loops.

A dimensional constant can not play the role of a small parameter and the expansion in \hbar is rather questionable. But it is easy to see that this expansion is actually well defined for the ϕ^4 model. To see this let us consider the path integral

$$Z = \int D[\phi] e^{-\int_x [\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4]} \quad (\text{B26})$$

and make the rescaling $\phi \rightarrow \phi/\sqrt{g}$,

$$Z = \int D[\phi] e^{-\frac{1}{g} \int_x [\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{1}{4!}\phi^4]}. \quad (\text{B27})$$

It shows that the perturbative limit $g \rightarrow 0$ suppresses all fluctuations of the rescaled field. Naturally, for small ϕ the quartic terms is negligible compared to the quadratic one and the perturbation expansion is applicable. Thus the perturbation expansion can be used to justify the loop-expansion.

We need the generating function

$$e^{\frac{1}{\hbar}W[j]} = \int D[\phi] e^{-\frac{1}{\hbar}S[\phi] + \frac{1}{\hbar}j \cdot \phi} \quad (\text{B28})$$

in the one-loop order which is the easiest to do by expanding the exponent of the integrand around the saddle point ϕ_0 , satisfying the equation of motion

$$\left. \frac{\delta S[\phi]}{\delta\phi} \right|_{\phi=\phi_0} = j. \quad (\text{B29})$$

The result is

$$e^{\frac{1}{\hbar}W[j]} = \int D[\phi] e^{-\frac{1}{\hbar}S[\phi] + \frac{1}{\hbar}j \cdot \phi - \frac{1}{2\hbar} \phi \cdot \frac{\delta^2 S[\phi_0]}{\delta\phi\delta\phi} \cdot \phi + \frac{1}{\hbar} \mathcal{O}(\phi^3)} \quad (\text{B30})$$

and

$$W[j] = -S[\phi_0] + j \cdot \phi_0 - \frac{\hbar}{2} \text{Tr} \ln \frac{\delta^2 S[\phi_0]}{\delta\phi\delta\phi} + \mathcal{O}(\hbar^2). \quad (\text{B31})$$

The leading order term for the new variable after the Legendre transformation

$$\begin{aligned} \Phi &= \frac{\delta W[j]}{\delta j} \\ &= -\frac{\delta S[\phi]}{\delta\phi} \cdot \frac{\delta\phi}{\delta j} \Big|_{\phi=\phi_0} + \phi_0 + j \cdot \frac{\delta\phi}{\delta j} \Big|_{\phi=\phi_0} + \mathcal{O}(\hbar) \\ &= \phi_0 + \mathcal{O}(\hbar) \end{aligned} \quad (\text{B32})$$

is the saddle point as expected and we find

$$\begin{aligned} \Gamma[\Phi] &= -W[j] + j \cdot \Phi \\ &= S[\Phi] + \frac{\hbar}{2} \text{Tr} \ln \frac{\delta^2 S[\phi]}{\delta\phi\delta\phi} \Big|_{\phi=\Phi} + \mathcal{O}(\hbar^2). \end{aligned} \quad (\text{B33})$$

The effective potential is defined by the effective action density for homogeneous field expectation value $\phi_x = \phi$,

$$U_{\text{eff}}(\phi) = \frac{1}{V} \Gamma[\phi] = U_B(\phi) + U^{(1)}(\phi) \quad (\text{B34})$$

where

$$U^{(1)}(\phi) = \frac{\hbar}{2} \int \frac{d^4 p}{(2\pi)^4} \ln[p^2 + U_B''(\phi)]. \quad (\text{B35})$$

The expansion of the logarithm yields

$$U^{(1)}(\phi) = -\frac{\hbar}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int \frac{d^4 p}{(2\pi)^4} \frac{U_B''(\phi)}{p^2 + U_B''(\phi)} \quad (\text{B36})$$

up to a field independent constant, the sum of the RPA ring graphs obtained by inserting the vertex $U_B''(\phi) = m_B^2 + g_B \phi^2/2$. Notice that the first two terms are divergent only and

$$U_F^{(1)}(\phi) = U^{(1)}(\phi) - \frac{1}{2} \phi^2 \partial_\phi^2 U^{(1)}(0) - \frac{1}{4!} \phi^4 \partial_\phi^4 U^{(1)}(0) \quad (\text{B37})$$

is finite. The subtracted part will be replaced by the renormalization conditions involving the physical, renormalized mass and coupling strength,

$$\begin{aligned} m^2 &= \partial_\phi^2 U_{\text{eff}}(0), \\ g &= \partial_\phi^4 U_{\text{eff}}(0). \end{aligned} \quad (\text{B38})$$

The integral

$$\int dx x \ln(x+a) = \frac{1}{2}(x^2 - a^2) \ln(x+a) - \frac{1}{4}x(x-2a) \quad (\text{B39})$$

gives

$$\begin{aligned} U^{(1)} &= \frac{\hbar\Omega_4}{2(2\pi)^4} \left[(\Lambda^4 - U_B''^2) \ln(\Lambda^2 + U_B'') - \frac{1}{2} \Lambda^2 (\Lambda^2 - 2U_B'') + U_B''^2 \ln U_B'' \right] \\ &= \frac{\hbar\Omega_4}{2(2\pi)^4} \left[\Lambda^4 \ln(\Lambda^2 + U_B'') - U_B''^2 \ln \frac{\Lambda^2 + U_B''}{U_B''} - \frac{1}{2} \Lambda^2 (\Lambda^2 - 2U_B'') \right] \\ &= \frac{\hbar}{4\pi^2} \left(2\Lambda^2 U_B'' - \frac{1}{2} U_B''^2 + U_B''^2 \ln \frac{U_B''}{\Lambda^2} \right) + \mathcal{O}(\Lambda^{-2}) \end{aligned} \quad (\text{B40})$$

up to a field independent constant for the momentum space cut-off Λ . The solid angle is $2\pi^2$ according to Eq. (C23).

To remove the divergences we add a counter terms

$$L_{CT} = \frac{\delta m^2}{2}\phi^2 + \frac{\delta g}{4!}\phi^4 \quad (\text{B41})$$

to the Lagrangian. The parameters

$$\begin{aligned} \delta m^2 &= -\partial_\phi^2 U^{(1)}(0) \\ \delta g &= -\partial_\phi^4 U^{(1)}(0) \end{aligned} \quad (\text{B42})$$

are chosen in such a manner that their leading order contributions in the loop-expansion carries out the subtraction (B37). The resulting is the adjustment of the mass and coupling constant of the Lagrangian and the renormalized, finite effective potential

$$U_{\text{eff } R}(\phi) = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + U_F^{(1)}(\phi). \quad (\text{B43})$$

The replacement $m_B^2 \rightarrow m^2$ and $g_B \rightarrow g$ is made within $U_F^{(1)}(\phi)$ because $m_B^2 = m^2(1 + \mathcal{O}(\hbar))$ and $g_B = g(1 + \mathcal{O}(\hbar))$.

The Taylor expansion of the effective potential in the field gives the 1PI vertex functions at vanishing momentum. Dimensional arguments show that the momentum dependence is finite on the one-loop order once the counterterms (B41) are used. From two-loop order we need an additional counterterm involving the wave function renormalization constant Z and the general form of the counterterm Lagrangian is

$$\frac{Z-1}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta g}{4!}\phi^4. \quad (\text{B44})$$

APPENDIX C: DIMENSIONAL REGULARIZATION

Some useful formulae are collected in this Appendix for dimensional regularization.

1. Γ -function

We start with the definition of the Γ -function:

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}, \quad \text{Re } z > 0 \quad (\text{C1})$$

what can be written as a sum of a regular and singular part,

$$\Gamma(z) = \underbrace{\int_0^1 dt e^{-t} t^{z-1}}_{P(z) \text{ singular}} + \underbrace{\int_1^\infty dt e^{-t} t^{z-1}}_{Q(z) \text{ regular}} \quad (\text{C2})$$

with

$$P(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 dt t^{k+z-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z}. \quad (\text{C3})$$

displaying a pole at negative integers.

The key property of the Γ function is the recursive relation

$$\Gamma(z+1) = z\Gamma(z) \quad (\text{C4})$$

which allows the identification

$$\Gamma(n+1) = n!. \quad (\text{C5})$$

because $\Gamma(1) = 1$.

Further useful relations involve the product of two neighboring values of the Γ -functions. We start with

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty ds dt e^{-s-t} t^{z-1} s^{-z}, \quad (\text{C6})$$

and change the variables $u = s + t$, and $v = \frac{t}{s}$ with $t = vs$, $s = u - vs$, $s = \frac{u}{1+v}$, $t = \frac{vu}{1+v}$,

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \int_0^\infty dudv \left| \begin{array}{cc} \frac{1}{1+v} & \frac{v}{1+v} \\ -\frac{u}{(1+v)^2} & \frac{(1+v)u-vu}{(1+v)^2} \end{array} \right| e^{-u} v^{z-1} \frac{1+v}{u} \\ &= \int_0^\infty dudv \frac{u+uv}{(1+v)^3} e^{-u} v^{z-1} \frac{1+v}{u} \\ &= \int_0^\infty dudv \frac{e^{-u} v^{z-1}}{1+v} \\ &= \int_0^\infty dv \frac{v^{z-1}}{1+v} \\ &= \frac{\pi}{\sin \pi z} \end{aligned} \quad (\text{C7})$$

Another useful formula is for

$$2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \int_0^\infty ds dt e^{-s-t} (2\sqrt{st})^{2z-1} t^{-1/2} \quad (\text{C8})$$

where the new variables $u = \sqrt{s}$, and $v = \sqrt{t}$ give

$$\begin{aligned} 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) &= 4 \int_0^\infty du dv e^{-u^2-v^2} (2uv)^{2z-1}, \quad (u \leftrightarrow v) \\ &= 2 \int_0^\infty dudv e^{-u^2-v^2} (2uv)^{2z-1} (u+v) \\ &= 4 \int_0^\infty du \int_0^u dv e^{-u^2-v^2} (2uv)^{2z-1} (u+v) \end{aligned} \quad (\text{C9})$$

We change variables again and use $x = u^2 + v^2$, and $y = 2uv$. The inverse relations $u = \frac{\sqrt{x+y} + \sqrt{x-y}}{2}$, $v = \frac{\sqrt{x+y} - \sqrt{x-y}}{2}$ give

$$\begin{aligned} 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) &= \int_0^\infty dx dy \frac{e^{-x} y^{2z-1}}{\sqrt{x-y}} \\ &= 2 \int_0^\infty d\bar{x} e^{-\bar{x}^2} \int_0^\infty dy e^{-y} y^{2z-1} \\ &= \sqrt{\pi} \Gamma(2z). \end{aligned} \quad (\text{C10})$$

This latter, taken at $z = \frac{1}{2}$ yields

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (\text{C11})$$

We need frequently the logarithmic derivative

$$\begin{aligned} \psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)} \\ &= -\frac{1}{z+n} + \underbrace{\Omega(z+n)}_{\text{regular}} \end{aligned} \quad (\text{C12})$$

which has similar singularity structure than the Γ -function and satisfies the simple relations

$$\begin{aligned}\psi(z+1) &= \frac{\Gamma(z) + z\Gamma'(z)}{z\Gamma(z)} = \frac{1}{z} + \psi(z) \\ \psi(1-z) - \psi(z) &= -\frac{\Gamma'(z)\Gamma(1-z) - \Gamma(z)\Gamma'(1-z)}{\Gamma(z)\Gamma(1-z)} \\ &= \pi \cot \pi z \\ \psi(z) + \psi\left(z + \frac{1}{2}\right) + 2\ln 2 &= 2\psi(2z).\end{aligned}\tag{C13}$$

The Euler's constant, defined by

$$\gamma = -\psi(1) = -\Gamma'(1)\tag{C14}$$

allows us to express the logarithmic derivatives,

$$\begin{aligned}\psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k}, \quad (z = \frac{1}{2}) \\ \psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln 2 \\ \psi\left(n + \frac{1}{2}\right) &= -\gamma - 2\ln 2 + 2\sum_{k=1}^n \frac{1}{2k-1}.\end{aligned}\tag{C15}$$

Important special cases, used in the calculation are

$$\begin{aligned}\Gamma(\epsilon) &= \frac{\Gamma(1+\epsilon)}{\epsilon} = \frac{\Gamma(1) + \epsilon\Psi(1)\Gamma(1)}{\epsilon} = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \\ \Gamma(-1+\epsilon) &= \frac{\Gamma(\epsilon)}{\epsilon-1} = -\frac{1}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon).\end{aligned}\tag{C16}$$

2. Loop-integrals

Our goal is to construct the analytic continuation of the loop-integral

$$I(\alpha, \beta, m^2) = \int d^d p \frac{(p^2)^\alpha}{(p^2 + m^2)^\beta}\tag{C17}$$

for complex dimensions. The integral is well defined for $\Re d < C$ where C a negative number and possesses a unique analytic structure on the d -plane.

In order to continue analytically the loop-integrals for complex dimensions we start with the integral

$$\int_0^\infty dt e^{-pt} t^{z-1} = \frac{\Gamma(z)}{p^z}\tag{C18}$$

and define the B -function

$$\begin{aligned}B(x, y) &= \int_0^1 dt t^{x-1} (1-t)^{y-1}, \quad \Re x, \Re y > 0, \quad u = \frac{t}{1-t} \\ &= \int_0^\infty du \frac{u^{x-1}}{(1+u)^{x+y}}\end{aligned}\tag{C19}$$

as the integral we wish to calculate. The relation

$$\frac{1}{(1+u)^{x+y}} = \frac{1}{\Gamma(x+y)} \int_0^\infty dt e^{-(1+u)t} t^{x+y-1}\tag{C20}$$

gives a representation of the B -function in terms of the Γ -functions,

$$\begin{aligned}
B(x, y) &= \frac{1}{\Gamma(x+y)} \int_0^\infty du u^{x-1} \int_0^\infty dt e^{-(1+u)t} t^{x+y-1} \\
&= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty dt e^{-t} t^{y-1} \\
&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\end{aligned} \tag{C21}$$

The last ingredient for the loop-integrals is the solid angle, Ω_d , obtained from the integral

$$\int d^d p e^{-p^2} = \Omega_d \int dp p^{d-1} e^{-p^2} = \frac{1}{2} \Omega_d \int dy y^{d/2-1} e^{-y} = \frac{1}{2} \Omega_d \Gamma\left(\frac{d}{2}\right) = \pi^{d/2} \tag{C22}$$

as

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}. \tag{C23}$$

We are finally in the position to calculate our loop-integral

$$\begin{aligned}
I(\alpha, \beta, m^2) &= \int d^d p \frac{(p^2)^\alpha}{(p^2 + m^2)^\beta} \\
&= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int dp \frac{(p^2)^{\alpha+\frac{d}{2}-\frac{1}{2}}}{(p^2 + m^2)^\beta} \\
&= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int dy \frac{y^{\alpha+\frac{d}{2}-1}}{(y + m^2)^\beta} \\
&= m^{2+2\alpha+d-2-2\beta} \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int dy \frac{y^{\alpha+\frac{d}{2}-1}}{(y + 1)^\beta} \\
&= \pi^{d/2} m^{2(\alpha-\beta)+d} \frac{\Gamma\left(\alpha + \frac{d}{2}\right) \Gamma\left(\beta - \alpha - \frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right) \Gamma(\beta)}
\end{aligned} \tag{C24}$$

Notice the peculiarity of dimensional regularization,

$$\int d^d p (p^2)^\alpha = 0. \tag{C25}$$

The special cases

$$\begin{aligned}
I_0(q, m^2) &= \int d^d p \frac{1}{(p^2 + 2pq + m^2)^\beta} \\
&= \int d^d p \frac{1}{(p^2 + m^2 - q^2)^\beta} \\
&= I(0, \beta, m^2 - q^2) \\
&= \pi^{d/2} (m^2 - q^2)^{d/2-\beta} \frac{\Gamma\left(\beta - \frac{d}{2}\right)}{\Gamma(\beta)},
\end{aligned} \tag{C26}$$

$$\begin{aligned}
I_\mu(q, m^2) &= \int d^d p \frac{p^\mu}{(p^2 + 2pq + m^2)^\beta} \\
&= \int d^d p \frac{p^\mu - q^\mu}{(p^2 + m^2 - q^2)^\beta} \\
&= -q^\mu I_0(q, m^2) \\
&= -q^\mu \pi^{d/2} (m^2 - q^2)^{d/2-\beta} \frac{\Gamma\left(\beta - \frac{d}{2}\right)}{\Gamma(\beta)},
\end{aligned} \tag{C27}$$

and

$$\begin{aligned}
I_{\mu\nu}(q, m^2) &= \int d^d p \frac{p^\mu p^\nu}{(p^2 + 2pq + m^2)^\beta} \\
&= \int d^d p \frac{(p^\mu - q^\mu)(p^\nu - q^\nu)}{(p^2 + m^2 - q^2)^\beta} \\
&= \int d^d p \frac{p^\mu p^\nu + q^\mu q^\nu}{(p^2 + m^2 - q^2)^\beta} \\
&= \int d^d p \frac{\delta^{\mu\nu} \frac{1}{d} p^2 + q^\mu q^\nu}{(p^2 + m^2 - q^2)^\beta} \\
&= \frac{1}{d} \delta^{\mu\nu} I(1, \beta, m^2 - q^2) + q^\mu q^\nu I_0(q) \\
&= \frac{1}{d} \delta^{\mu\nu} \pi^{d/2} (m^2 - q^2)^{1-\beta+d/2} \frac{\Gamma(1 + \frac{d}{2}) \Gamma(\beta - 1 - \frac{d}{2})}{\Gamma(\frac{d}{2}) \Gamma(\beta)} + q^\mu q^\nu \pi^{d/2} (m^2 - q^2)^{d/2-\beta} \frac{\Gamma(\beta - \frac{d}{2})}{\Gamma(\beta)} \\
&= \frac{\delta^{\mu\nu}}{d} \pi^{d/2} (m^2 - q^2)^{1-\beta+d/2} \frac{\frac{d}{2} \Gamma(\frac{d}{2}) \Gamma(\beta - \frac{d}{2})}{(\beta - 1 - \frac{d}{2}) \Gamma(\frac{d}{2}) \Gamma(\beta)} + q^\mu q^\nu \pi^{d/2} (m^2 - q^2)^{d/2-\beta} \frac{\Gamma(\beta - \frac{d}{2})}{\Gamma(\beta)} \\
&= \left[\frac{\delta^{\mu\nu}}{2} (m^2 - q^2) \frac{1}{\beta - 1 - \frac{d}{2}} + q^\mu q^\nu \right] \underbrace{\pi^{d/2} (m^2 - q^2)^{d/2-\beta} \frac{\Gamma(\beta - \frac{d}{2})}{\Gamma(\beta)}}_{I_0(q, m^2)} \tag{C28}
\end{aligned}$$

are particularly useful for low-order graphs.

The question of continuing the Dirac-matrix algebra in the space-time dimensions is highly non-trivial due to the axial anomaly and is not addressed here.